LIMIT HEIGHTS AND SPECIAL VALUES OF THE RIEMANN ZETA FUNCTION

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Abstract. We study the distribution of the height of the intersection between the projective line defined by the linear polynomial $x_0 + x_1 + x_2$ and its translate by a torsion point. We show that for a strict sequence of torsion points, the corresponding heights converge to a real number that is a rational multiple of a quotient of special values of the Riemann zeta function. We also determine the range of these heights, characterize the extremal cases, and study their limit for sequences of torsion points that are strict in proper algebraic subgroups.

In addition, we interpret our main result from the viewpoint of Arakelov geometry, showing that for a strict sequence of torsion points the limit of the corresponding heights coincides with an Arakelov height of the cycle of the projective plane over the integers defined by the same linear polynomial. This is a particular case of a conjectural asymptotic version of the arithmetic Bézout theorem.

Using the interplay between arithmetic and convex objects from the Arakelov geometry of toric varieties, we show that this Arakelov height can be expressed as the mean of a piecewise linear function on the amoeba of the projective line, which in turn can be computed as the aforementioned real number.

Contents

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Introduction

Solving systems of polynomial equations in several variables is one of the guiding problems of mathematics, and has motivated the rise of linear algebra and of algebraic geometry, other than being applied to disparate areas across the sciences.

While solving a given system of polynomial equations is a business of algorithmic research, a more theoretical approach is concerned with obtaining the finest possible information about its solution set in terms of data that can be read directly from the system. Basically, this perspective aims at understanding the complexity of the solution set in terms of the complexity of the system.

From this point of view, the prototypical result is the classical Bézout's theorem, asserting that for a generic system of n-many homogeneous polynomials in $n+1$ variables, the cardinality of its solution set in the n-dimensional projective space equals the product of the degrees of these polynomials.

For systems whose coefficients are algebraic numbers we can consider not only the geometric complexity of its solution set given by its cardinality, but also its arithmetic complexity. This latter is usually defined as the maximal bit-length of the integers in a representation of the solution set and can be measured in terms of its height, see for instance Remark 1.1.

In this context, it is then natural to ask whether the height of the solution set can be predicted from the arithmetic complexity of the system, measured for example in terms of the height of the involved polynomials. The goal of this article is to investigate this question through the study of an explicit example.

Let us now set our playground more precisely. Let $\mathbb{P}^2(\overline{\mathbb{Q}})$ be the projective plane over the algebraic closure of the field of rational numbers and consider its (canonical) height function, that is the real-valued function

$$\mathrm{h}\colon \mathbb{P}^2(\overline{\mathbb{Q}}) \longrightarrow \mathbb{R}$$

introduced and studied by Northcott and Weil [Nor50, Wei51], see also Section 1. For ω_1, ω_2 varying in the set of roots of unity $\mu_\infty \subset \overline{\mathbb{Q}}$, put $\omega = (\omega_1, \omega_2)$ and consider the system of linear equations on $\mathbb{P}^2(\overline{\mathbb{Q}})$ given as

$$x_0 + x_1 + x_2 = x_0 + \omega_1^{-1} x_1 + \omega_2^{-1} x_2 = 0. \tag{1}$$

Apart from the degenerate case $\omega_1 = \omega_2 = 1$ which we exclude throughout, its solution set consists of a single point, that we denote by $P(\omega)$.

From a geometrical perspective, note that the (algebraic) torus $\mathbb{G}_{\mathrm{m}}^2(\overline{\mathbb{Q}}) = (\overline{\mathbb{Q}}^\times)^2$ has a standard action on $\mathbb{P}^2(\overline{\mathbb{Q}})$ as explained in (1.3), and that the two equations in (1) correspond respectively to the projective line $C = Z(x_0 + x_1 + x_2) \subset \mathbb{P}^2(\overline{\mathbb{Q}})$ and to its translate $\omega\, C$ by the torsion point ω of this torus. The intersection of these two lines coincides with the solution set of the system of equations in (1) and hence, when ω is nontrivial,

$$C \cap \omega\, C = \{P(\omega)\}.$$

Our aim is to understand how the height of the intersection $C \cap \omega\, C$, or equivalently that of the point $P(\omega)$, depends on the choice of ω. The first observation is that one cannot determine it from the usual measures for the complexity of the system. Indeed any such measure, like the degrees or the Newton polytopes of the defining polynomials, their Mahler measures or any norm depending only on the absolute

values of the coefficients, is constant as ω varies. On the contrary, different choices of the torsion point ω can produce projective points $P(\omega)$ with very different heights [Gua18a, Example 5.1.1], see also Example 2.1 or Figure 1 in this introduction.

In view of this situation, we turn to the study of the distribution of the height of $P(\omega)$, starting by determining its range of values.

Proposition 1 (Proposition 2.2)**.** *Let $\omega \in \mathbb{G}_{\mathrm{m}}^2(\overline{\mathbb{Q}})$ be a nontrivial torsion point. Then*

$$0 \leq \mathrm{h}(C \cap \omega\, C) \leq \log(2).$$

More precisely, we show that these bounds are sharp and characterize the cases in which each of them is attained. These results follow from an explicit expression for the projective point $P(\omega)$, the basic properties of the height, and a formula for the value of a cyclotomic polynomial evaluated at 1.

Having established the range of these height values, the next question is to understand how they distribute within the interval $[0, \log(2)]$. We can gain insight into it by means of numerical experimentation, as we detail in Section 8 and in the accompanying `SageMath` notebook [GS22].

In practice, let d be a positive integer and denote by μ_d the set of d-roots of unity, so that μ_d^2 coincides with the set of d-torsion points of $\mathbb{G}_{\mathrm{m}}^2(\overline{\mathbb{Q}})$. Subdividing the unit square into d^2-many cells, we can assign to each of them a d-torsion point $\omega \in \mu_d^2$ and, excluding the case when ω is trivial, color it with a tone of gray that is as dark as the height of $P(\omega)$ is large within the range $[0, \log(2)]$. As shown in Figure 1, these tones distribute on the square without following any clear pattern.

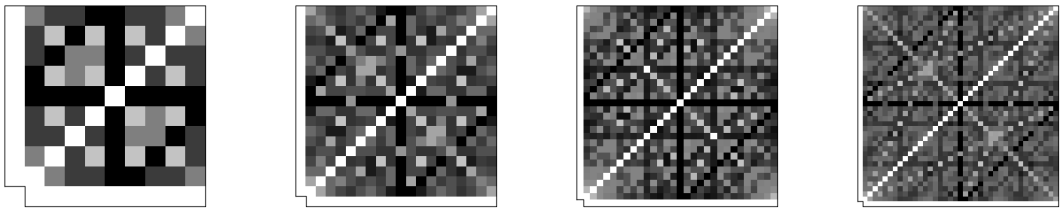

FIGURE 1. Heights associated to nontrivial d-torsion points for $d = 10, 20, 30, 40$

However, a careful look at these images reveals an interesting phenomenon: as d grows bigger, most of the cells get colored with a very similar tone of gray, suggesting that most of the corresponding heights are close to a precise real number. This becomes even more evident for higher values of d, such as those in Figures 8.1 and 8.3.

Guided by this intuition, we focus on the study of the height of $P(\omega)$ for a generic torsion point ω, that is for ω varying in a generic sequence of torsion points. Because of the former toric Manin–Mumford conjecture, the latter is nothing else but a *strict* sequence, that is a sequence which eventually avoids any fixed proper algebraic subgroup of $\mathbb{G}_{\mathrm{m}}^2(\overline{\mathbb{Q}})$. The following is the main result of the article.

Theorem 2 (Theorem 6.1)**.** *Let $(\omega_\ell)_{\ell \geq 1}$ be a strict sequence in $\mathbb{G}_{\mathrm{m}}^2(\overline{\mathbb{Q}})$ of nontrivial torsion points. Then*

$$\lim_{\ell \to +\infty} \mathrm{h}(C \cap \omega_\ell C) = \frac{2\,\zeta(3)}{3\,\zeta(2)} = 0.487175\ldots,$$

where ζ is the Riemann zeta function.

This appearance of the Riemann zeta function is intriguing. However, there are already several known relations between heights and special values of L-functions. Just to cite a few, akin links are given by the Gross–Zagier formula for the height of Heegner points on modular curves [GZ86] and by the fulfillment of similar properties like the Northcott one [PP20]. Moreover, such connections have motivated far-reaching conjectures, like that by Colmez [Col93] and its generalization by Maillot and Rössler [MR02].

Closer to our setting, the height of projective points can be extended to projective subvarieties. In the particular case of a projective hypersurface defined over $\mathbb{Q}$, this notion coincides with the logarithmic Mahler measure of a primitive defining polynomial [DP99, Mai00], and there is an active line of research relating Mahler measures with special values of L-functions, see for instance [BZ20]. For example, it follows from a classical result of Smyth [Smy81] that the height of the projective line C can be computed as

$$\mathrm{h}(C) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) = 0.323065\ldots$$

for the L-function corresponding to the odd Dirichlet character modulo 3. Indeed, this value also shows up in our investigations as the limit of the height of $P(\omega)$ for ω varying in a strict sequence of torsion points of a certain 1-dimensional algebraic subtorus (Example 7.5) and as the conjectural minimal accumulation value of the set of heights of $P(\omega)$ for $\omega \in \mu_\infty^2$ (Question 8.3).

A formal argument allows to deduce from Theorem 2 the following result, which justifies our previous observation from the numerical experiments.

Corollary 3 (Corollary 6.8)**.** *For each $\varepsilon > 0$ we have that*

$$\lim_{d\to+\infty} \frac{1}{d^2-1} \#\Big\{\omega \in \mu_d^2 \setminus \{(1,1)\} \,\Big|\, \Big|\mathrm{h}(C\cap\omega C) - \frac{2\,\zeta(3)}{3\,\zeta(2)}\Big| < \varepsilon\Big\} = 1.$$

Loosely speaking, it asserts that the *typical value* of the heights corresponding to d-torsion points is the mentioned rational multiple of a quotient of special values of the Riemann zeta function. Actually, this property holds in greater generality for strict sequences of finite subsets of torsion points (Theorem 6.6).

Let us now outline the proof of Theorem 2. First we write the height corresponding to a nontrivial torsion point $\omega = (\omega_1, \omega_2)$ of order d as the sum

$$\mathrm{h}(P(\omega)) = \sum_v \frac{1}{\varphi(d)} \sum_{k\in(\mathbb{Z}/d\,\mathbb{Z})^\times} \log\max(|\iota_v(\omega_2^k) - \iota_v(\omega_1^k)|_v, |\iota_v(\omega_2^k) - 1|_v, |\iota_v(\omega_1^k) - 1|_v). \tag{2}$$

Here $(\mathbb{Z}/d\,\mathbb{Z})^\times$ denotes the group of modular units and φ the Euler totient function, whereas v ranges over the set of places of $\mathbb{Q}$ and $\iota_v\colon \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_v$ is any embedding of $\overline{\mathbb{Q}}$ into the algebraically closed complete field of v-adic numbers.

In slightly more sophisticated words, the summand corresponding to a place v in the previous height formula can be viewed as the mean over the v-adic Galois orbit of ω of a certain function on the v-adic torus $\mathbb{G}_{\mathrm{m}}^2(\mathbb{C}_v) = (\mathbb{C}_v^\times)^2$ having a logarithmic singularity at the point $(1,1)$. Hence to compute the limit of this height for ω going over a strict sequence of torsion points, we need to prove an equidistribution result for an adelic family of functions with logarithmic singularities, with a simultaneous control of all the v-adic summands.

In our concrete situation, we achieve this by following an elementary approach. On the one hand, we show that the sum of the non-Archimedean summands in (2) can be computed in terms of the von Mangoldt and the Euler totient functions (Corollary 3.4). This implies that for a sequence of torsion points with diverging order, the non-Archimedean contribution to the height converges to 0 (Corollary 3.6).

On the other hand, the convergence of the Archimedean summand can be deduced from either the logarithmic equidistribution theorem of Chambert-Loir and Thuillier [CT09] or from that of Dimitrov and Habegger [DH19]. However, it is also possible to proceed more directly and deduce this convergence from basic results about cyclotomic polynomials and the standard equidistribution of Galois orbits of torsion points of tori (Proposition 4.1). In any case, we obtain that for a strict sequence of torsion points, the Archimedean summand in (2) tends to the integral

$$\frac{1}{(2\pi)^2}\int_{(\mathbb{R}/2\pi\mathbb{Z})^2} \log\max(|\mathrm{e}^{\mathrm{i}u_2}-\mathrm{e}^{\mathrm{i}u_1}|, |\mathrm{e}^{\mathrm{i}u_2}-1|, |\mathrm{e}^{\mathrm{i}u_1}-1|)\, du_1 du_2.$$

Taking advantage of the symmetries of the integrand, we compute it as the stated quotient involving special values of the Riemann zeta function (Proposition 5.1), thus completing the proof.

The results presented so far represent the content of Part I of the article. The approach therein is explicit and the arguments employed are as self-contained as possible. This whole part requires very little background, with the purpose of making it accessible to non-experts.

In Part II we raise the technical level of the exposition to present an interpretation of Theorem 2 from the viewpoint of Arakelov geometry, as developed by Gillet and Soulé [GS90] and extended by Maillot [Mai00]. This allows to recover it in a more intrinsic way through the interplay between arithmetic and convex objects from the Arakelov geometry of toric varieties, studied by Burgos Gil, Philippon and the second named author [BPS14] and by the first named author [Gua18b].

Arakelov geometry provides a vast generalization of the notion of height, from projective points to cycles of an arithmetic variety equipped with a family of metrized line bundles. In this context, the height of a point of $\mathbb{P}^2(\overline{\mathbb{Q}})$ with rational coordinates coincides with the (Arakelov) height of its associated 1-dimensional subscheme of the projective plane over the integers $\mathbb{P}^2_{\mathbb{Z}}$ with respect to the canonical metrized line bundle $\overline{\mathscr{O}(1)}^{\mathrm{can}}$.

Another distinguished metrized line bundle on $\mathbb{P}^2_{\mathbb{Z}}$ is the Ronkin metrized line bundle $\overline{\mathscr{O}(1)}^{\mathrm{Ron}}$, constructed from the Ronkin function of $x_0+x_1+x_2$. It is relevant in the study of the Arakelov geometry of the hypersurface $\mathscr{C}$ of $\mathbb{P}^2_{\mathbb{Z}}$ defined by this linear polynomial [Gua18b].

Since these two metrized line bundles are semipositive, arithmetic intersection theory allows to define the height

$$\mathrm{h}_{\overline{\mathscr{O}(1)}^{\mathrm{can}},\,\overline{\mathscr{O}(1)}^{\mathrm{Ron}}}(\mathscr{C}) \in \mathbb{R}. \tag{3}$$

As a consequence of the results in Part I and the metric Weil reciprocity law (Proposition 9.6), we can show that the limit value in Theorem 2 coincides with this height.

Theorem 4 (Theorem 10.4)**.** *Let $(\omega_\ell)_{\ell\geq 1}$ be a strict sequence in $\mathbb{G}_{\mathrm{m}}^2(\overline{\mathbb{Q}})$ of nontrivial torsion points. Then*

$$\lim_{\ell\to+\infty} \mathrm{h}(C\cap\omega_\ell C) = \mathrm{h}_{\overline{\mathscr{O}(1)}^{\mathrm{can}},\overline{\mathscr{O}(1)}^{\mathrm{Ron}}}(\mathscr{C}).$$

We can reformulate this result in the following suggestive form. Let $\mathbb{P}^2_{\overline{\mathbb{Z}}}$ be the projective plane over the integral closure of the integers. For each $\ell\geq 1$ write $\omega_\ell = (\omega_{\ell,1},\omega_{\ell,2})$ with $\omega_{\ell,1},\omega_{\ell,2}\in\mu_\infty$ and consider the 1-dimensional integral subscheme

$$Z(x_0+x_1+x_2, x_0+\omega_{\ell,1}^{-1}x_1+\omega_{\ell,2}^{-1}x_2)\subset\mathbb{P}^2_{\overline{\mathbb{Z}}}.$$

It coincides with the closure of $P(\omega_\ell)$ in $\mathbb{P}^2_{\overline{\mathbb{Z}}}$, and so its height with respect to $\overline{\mathscr{O}(1)}^{\mathrm{can}}$ agrees with the height of the point $P(\omega_\ell)$ (Remark 10.3).

The main result of [Gua18b] shows that the height in (3) coincides with the height of the ambient projective plane with respect to a further Ronkin metrized line bundle (Proposition 11.6). Combining these results yields the following one.

Corollary 5 (Corollary 11.7)**.** *With notation as in Theorem 4,*

$$\lim_{\ell\to+\infty} \mathrm{h}_{\overline{\mathscr{O}(1)}^{\mathrm{can}}}(Z(x_0+x_1+x_2, x_0+\omega_{\ell,1}^{-1}x_1+\omega_{\ell,2}^{-1}x_2)) = \mathrm{h}_{\overline{\mathscr{O}(1)}^{\mathrm{can}},\overline{\mathscr{O}(1)}^{\mathrm{Ron}},\overline{\mathscr{O}(1)}^{\mathrm{Ron}}}(\mathbb{P}^2_{\mathbb{Z}}).$$

This limit formula can be considered as a particular case of a conjectural arithmetic analogue of the classical geometric fact that for a family of n-many line bundles on an n-dimensional algebraic variety, the cardinality of the zero set of a generic n-tuple of their global sections coincides with the degree of the variety with respect to these line bundles. Indeed, Corollary 5 shows that a height of the zero set of a pair of global sections of $\mathscr{O}(1)$ with a certain arithmetic feature approaches a related height of the ambient space as these global sections becomes more and more "generic".

Let us give some hints on how to recover Theorem 2 from Theorem 4 through the Arakelov geometry of toric arithmetic varieties, applied to the case of $\mathbb{P}^2_{\mathbb{Z}}$. A fundamental ingredient of this theory is the classification of semipositive toric metrized line bundles in terms of certain concave functions on a vector space [BPS14]. Within this classification, the metrized line bundle $\overline{\mathscr{O}(1)}^{\mathrm{can}}$ on $\mathbb{P}^2_{\mathbb{Z}}$ corresponds to the piecewise linear concave function $\Psi\colon\mathbb{R}^2\to\mathbb{R}$ defined as

$$\Psi(u_1,u_2)=\min(0,u_1,u_2).$$

On the other hand, we associate to C its Archimedean amoeba $\mathscr{A}$, that is the tentacle-shaped subset of $\mathbb{R}^2$ given as the tropicalization the complex line $C(\mathbb{C})$ (Figure 12.1).

Applying the results of [BPS14, Gua18b] we can express the height in (3) in terms of convex geometry (Proposition 11.6). Combining this with the results of Passare and Rullgård on Ronkin functions and their associated Monge–Ampère measures [PR04], we deduce the following relation between the considered height and the average of the concave function Ψ on the amoeba $\mathscr{A}$.

Theorem 6 (Corollary 13.3)**.** *With notation as above,*

$$\mathrm{h}_{\overline{\mathscr{O}(1)}^{\mathrm{can}},\overline{\mathscr{O}(1)}^{\mathrm{Ron}}}(\mathscr{C}) = -\frac{1}{\mathrm{vol}(\mathscr{A})}\int_{\mathscr{A}}\Psi(u_1,u_2)\ du_1du_2.$$

Finally we compute the integral in this equality (Proposition 12.2) and thus recover Theorem 2 from Theorems 4 and 6.

Summing up, this article deals with a specific problem that is both suitable for a down-to-earth analysis and for numerical experimentation, as well as appearing as an instance of a much more general situation.

The concreteness of the results in Part I allows for an elementary and self-contained treatment that requires little background in algebraic geometry and number theory. Moreover, the considered problem is well-suited for computations that allow to visualize the results and to suggest further questions, as done in Section 8.

Placing the problem within the context of Arakelov geometry as in Part II, our investigation takes a deeper connotation, pointing towards an asymptotic version of the arithmetic Bézout theorem. Indeed, after Corollary 5 it seems reasonable to expect that the limit of the height of the solution set of a system of polynomial equations sharing a certain arithmetic feature coincides with a height of the ambient space, see for instance Conjecture 11.8. Establishing this extension would require a substantial technical effort to solve the adelic logarithmic equidistribution problem that arises. We plan to address this in a subsequent article.

In a more speculative spirit, it would be also interesting to explore whether some of these toric heights can be linked to integrals of piecewise linear functions over amoebas like in Theorem 6, and try to compute them in terms of special functions.

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PART I

This part is dedicated to the proof of our main results. We start in Section 1 by setting the notation and recalling the principal actors of our statements, like strict sequences of torsion points of algebraic tori and the height of a projective point.

In Section 2 we give a sharp bound for the height of the intersection of the projective line defined by the linear polynomial $x_0 + x_1 + x_2$ and its translate by a nontrivial torsion point.

The following four sections culminate with Theorem 6.1, which shows that the limit of such heights for torsion points in a strict sequence can be computed in terms of special values of the Riemann zeta function. In Section 6 we also extend this result to strict subsets of torsion points, to show that most of the corresponding heights concentrate near that limit value.

The treatment throughout is down-to-earth and as self-contained as possible, except maybe for Section 7, which might be skipped at a first read. Its goal is to extend the previous study to torsion points in proper subgroups of the 2-dimensional torus.

Finally, Section 8 illustrates the results through numerical calculations and graphical plottings made with the `SageMath` notebook [GS22] accompanying this article.

1. Preliminaries

Here we discuss some of the basic constructions and properties concerning algebraic tori and canonical heights on projective spaces. Our treatment is far from being complete, and we refer the interested reader to [BG06, Chapters 1 and 3] for the proofs and more details about the explained facts.

We denote by $\overline{\mathbb{Q}}$ an algebraic closure of the field of rational numbers $\mathbb{Q}$. For an integer $d \geq 1$ we write μ_d for the subgroup of $\overline{\mathbb{Q}}^\times$ of d-roots of unity, and μ_d° for its subset of primitive d-roots. We also denote by μ_∞ the subgroup of $\overline{\mathbb{Q}}^\times$ of all roots of unity.

For $n \geq 0$ we denote by $\mathbb{G}_{\mathrm{m}}^n(\overline{\mathbb{Q}}) = (\overline{\mathbb{Q}}^\times)^n$ the n-dimensional (split, algebraic) torus over $\overline{\mathbb{Q}}$. It is a group under coordinate-wise multiplication, with torsion subgroup equal to μ_∞^n. For $d \geq 1$ and $\zeta \in \mu_d^\circ$, each d-torsion point $\omega \in \mu_d^n$ of this torus can be uniquely written as

$$\omega = (\zeta^{c_1}, \dots, \zeta^{c_n}) \tag{1.1}$$

with $c_i \in \{0, \dots, d-1\}$ for $i = 1, \dots, n$. Its order is $\operatorname{ord}(\omega) = d/\gcd(c_1, \dots, c_n, d)$.

Given an algebraic subset V of $\mathbb{G}_{\mathrm{m}}^n(\overline{\mathbb{Q}})$, a sequence $(\gamma_\ell)_{\ell \geq 1}$ in V is *strict* if it eventually avoids any fixed algebraic subgroup H of this torus not containing V, that is, if there is $\ell_0 \geq 1$ such that $\gamma_\ell \notin H$ for all $\ell \geq \ell_0$. Our main case of interest is $V = \mathbb{G}_{\mathrm{m}}^n(\overline{\mathbb{Q}})$.

For each vector $a = (a_1, \dots, a_n) \in \mathbb{Z}^n$ we denote by χ^a the corresponding character, so that for $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{G}_{\mathrm{m}}^n(\overline{\mathbb{Q}})$ we have that

$$\chi^a(\gamma) = \gamma_1^{a_1} \cdots \gamma_n^{a_n}.$$

Any proper algebraic subgroup of $\mathbb{G}_{\mathrm{m}}^n(\overline{\mathbb{Q}})$ is contained in an algebraic subgroup of codimension 1, which are those defined by a binomial of the form $\chi^a - 1$ with $a \in \mathbb{Z}^n \setminus \{(0, \dots, 0)\}$. Hence a sequence $(\gamma_\ell)_{\ell \geq 1}$ in $\mathbb{G}_{\mathrm{m}}^n(\overline{\mathbb{Q}})$ is strict if and only if for every such a there is $\ell_0 \geq 1$ with

$$\chi^a(\gamma_\ell) \neq 1 \quad \text{for all } \ell \geq \ell_0. \tag{1.2}$$

We denote by $\mathbb{P}^n(\overline{\mathbb{Q}})$ the n-dimensional projective space over $\overline{\mathbb{Q}}$. For a homogeneous polynomial $f \in \overline{\mathbb{Q}}[x_0, \dots, x_n]$ we denote by $Z(f)$ the algebraic subset of this projective space that it defines, and for $\gamma \in \mathbb{G}_{\mathrm{m}}^n(\overline{\mathbb{Q}})$ we consider the *twist* of f by γ, which is the homogeneous polynomial $\gamma^* f \in \overline{\mathbb{Q}}[x_0, \dots, x_n]$ defined as

$$\gamma^* f(x_0, \dots, x_n) = f(x_0, \gamma_1 x_1, \dots, \gamma_n x_n).$$

For $\gamma \in \mathbb{G}_{\mathrm{m}}^n(\overline{\mathbb{Q}})$ we also consider the associated *translation map*

$$\mathbb{P}^n(\overline{\mathbb{Q}}) \longrightarrow \mathbb{P}^n(\overline{\mathbb{Q}}), \quad \xi = [\xi_0 : \xi_1 : \cdots : \xi_n] \longmapsto \gamma\,\xi := [\xi_0 : \gamma_1\,\xi_1 : \cdots : \gamma_n\,\xi_n]. \tag{1.3}$$

The translation by γ of the zero set of f coincides with the zero set of the twist $(\gamma^{-1})^* f$, that is

$$\gamma Z(f) = Z((\gamma^{-1})^* f)). \tag{1.4}$$

We next recall the basic notations and properties of canonical heights in projective spaces. We denote by $M_{\mathbb{Q}}$ the set of places of $\mathbb{Q}$, that is the set of equivalence classes of nontrivial absolute values on this field with respect to the topology that they define. By Ostrowski's theorem, these places are represented by the usual Archimedean absolute value on $\mathbb{Q}$ and by the p-adic ones as p ranges over the primes of $\mathbb{Z}$, and so $M_{\mathbb{Q}}$ can be identified with the set made of the symbol ∞ and these primes. For each $v \in M_{\mathbb{Q}}$ we denote by $|\cdot|_v$ its representative, and by $\mathbb{Q}_v$ the completion of $\mathbb{Q}$ with respect to this absolute value. When $v = \infty$ this complete field coincides with the field of real numbers $\mathbb{R}$, whereas when $v = p$ is a prime it is the field of p-adic numbers.

More generally, for a number field K we denote by M_K the set of its places. For each $w \in M_K$ there is a unique $v \in M_{\mathbb{Q}}$ such that $|\cdot|_v$ extends to a (unique) absolute value on K in the equivalence class of w, a relation that is indicated by $w \mid v$. We denote by $|\cdot|_w$ this absolute value on K, and by K_w the corresponding completion of K. For each $v \in M_{\mathbb{Q}}$, the set of places of K extending v is finite and moreover the sum of the corresponding local degrees coincides with the degree of the extension:

$$\sum_{w|v} [K_w : \mathbb{Q}_v] = [K : \mathbb{Q}]. \tag{1.5}$$

Let now $\xi = [\xi_0 : \cdots : \xi_n] \in \mathbb{P}^n(\overline{\mathbb{Q}})$. For each $v \in M_{\mathbb{Q}}$, the *v-adic height* of the vector of homogeneous coordinates of this projective point is defined as

$$\mathrm{h}_v(\xi_0, \ldots, \xi_n) = \sum_{w|v} \frac{[K_w : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log \max(|\xi_0|_w, \ldots, |\xi_n|_w)$$

for any number field K containing all these coordinates. Its value does not depend on the choice of this number field, and it vanishes for all but a finite number of v's. The *(canonical) height* of ξ is defined as the sum of these local heights:

$$\mathrm{h}(\xi) = \sum_{v \in M_{\mathbb{Q}}} \mathrm{h}_v(\xi_0, \ldots, \xi_n). \tag{1.6}$$

Thanks to the product formula, its value does not depend on the choice of homogeneous coordinates.

In general $\mathrm{h}(\xi) \geq 0$ and, by Kronecker's theorem, $\mathrm{h}(\xi) = 0$ if and only if the point can be written as $\xi = [\xi_0 : \cdots : \xi_n]$ with ξ_j equal to either 0 or a root of unity.

Remark 1.1. The height of a projective point is a measure of the complexity of its representation. For instance, a point $\xi \in \mathbb{P}^n(\overline{\mathbb{Q}})$ with rational homogeneous coordinates can be written as $\xi = [\xi_0 : \cdots : \xi_n]$ with coprime integer ξ_i's. In this situation, the formula in (1.6) boils down to

$$\mathrm{h}(\xi) = \log \max(|\xi_0|_\infty, \ldots, |\xi_n|_\infty),$$

which gives the maximal bit-length of these integers.

We next give a Galois-theoretic formula for the local heights of the vector of homogeneous coordinates of a projective point. Set $\Xi = (\xi_0, \ldots, \xi_n) \in \overline{\mathbb{Q}}^{n+1}$ and let

$$O(\Xi) = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot \Xi$$

be the orbit of this vector under the coordinate-wise action of the absolute Galois group of $\mathbb{Q}$. It is a finite subset of $\overline{\mathbb{Q}}^{n+1}$.

For each $v \in M_{\mathbb{Q}}$ we choose an algebraic closure of the complete field $\mathbb{Q}_v$, and we denote by $\mathbb{C}_v$ its completion with respect to the unique extension of $|\cdot|_v$ to it. This

field is both algebraically closed and complete with respect to the induced absolute value, that we also denote by $|\cdot|_v$.

We also choose an embedding

$$\iota_v \colon \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_v, \tag{1.7}$$

which induces an embedding $\overline{\mathbb{Q}}^{n+1} \hookrightarrow \mathbb{C}_v^{n+1}$ that we denote with the same symbol. The *v-adic Galois orbit* of Ξ is then defined as the image of $O(\Xi)$ under it, namely

$$O(\Xi)_v = \iota_v(O(\Xi)).$$

It is a finite subset of $\mathbb{C}_v^{n+1}$ with the same cardinality of $O(\Xi)$, and which does not depend on the choice of ι_v.

Proposition 1.2. *Let $\xi \in \mathbb{P}^n(\overline{\mathbb{Q}})$ and $\Xi \in \overline{\mathbb{Q}}^{n+1}$ a corresponding vector of homogeneous coordinates. Then for each $v \in M_{\mathbb{Q}}$*

$$\mathrm{h}_v(\Xi) = \frac{1}{\#O(\Xi)_v} \sum_{\Lambda \in O(\Xi)_v} \log\max(|\Lambda_0|_v, \dots, |\Lambda_n|_v)$$

and so

$$\mathrm{h}(\xi) = \frac{1}{\#O(\Xi)} \sum_{v \in M_{\mathbb{Q}}} \sum_{\Lambda \in O(\Xi)_v} \log\max(|\Lambda_0|_v, \dots, |\Lambda_n|_v).$$

In particular, the height is invariant under the action of the absolute Galois group of $\mathbb{Q}$.

Proof. Let $K \subset \overline{\mathbb{Q}}$ be a finite Galois extension of $\mathbb{Q}$ containing all the coordinates of Ξ, and denote by G its Galois group. For $v \in M_{\mathbb{Q}}$ denote by $M_{K,v}$ the set of places of K extending v, and by $w_0 \in M_{K,v}$ the place represented by the absolute value on K induced by the absolute value of $\mathbb{C}_v$ through the embedding ι_v in (1.7).

The group G has an action on the finite set $M_{K,v}$, that can be defined by considering for each pair $\sigma \in G$ and $w \in M_{K,v}$ the place $\sigma(w) \in M_{K,v}$ represented by the absolute value on K given by

$$|\alpha|_{\sigma(w)} = |\sigma(\alpha)|_w \quad \forall \alpha \in K.$$

By [Neu99, Chapter II, Proposition 9.1], this action is transitive.

The $\mathbb{Q}$-automorphism $\sigma \colon K \to K$ extends to a $\mathbb{Q}_v$-isomorphism $K_w \to K_{\sigma(w)}$, and so $[K_{\sigma(w)} : \mathbb{Q}_v] = [K_w : \mathbb{Q}_v]$. Since the action is transitive, the local degrees corresponding to the places in $M_{K,v}$ coincide. By the formula in (1.5), this implies that

$$\frac{[K_w : \mathbb{Q}_v]}{[K : \mathbb{Q}]} = \frac{1}{\#M_{K,v}}.$$

Moreover, by the orbit-stabilizer theorem one also has that $\#M_{K,v} = \#G/\#G_{w_0}$ where G_{w_0} denotes the stabilizer of the place w_0. Writing $\Xi = (\xi_0, \dots, \xi_n)$ we have

$$\mathrm{h}_v(\Xi) = \sum_{w \in M_{K,v}} \frac{1}{\#M_{K,v}} \log\max_j |\xi_j|_w = \frac{1}{\#G} \sum_{\sigma \in G} \log\max_j |\xi_j|_{\sigma(w_0)}.$$

On the other hand, G also acts transitively on the Galois orbit of Ξ and so

$$\frac{1}{\#O(\Xi)_v} = \frac{1}{\#O(\Xi)} = \frac{\#G_\Xi}{\#G},$$

where G_Ξ is the stabilizer of this vector. Hence

$$\frac{1}{\#G}\sum_{\sigma\in G}\log\max_j|\xi_j|_{\sigma(w_0)} = \frac{1}{\#G}\sum_{\sigma\in G}\log\max_j|(\iota_v\circ\sigma)(\xi_j)|_v$$
$$= \frac{1}{\#O(\Xi)_v}\sum_{\Lambda\in O(\Xi)_v}\log\max_j|\Lambda_j|_v,$$

proving the first statement. The other claims follow directly. □

Remark 1.3. A version of this result in the more general setting of Arakelov geometry appears in [BPRS19, Proposition 2.3].

2. The range of the height

In this section we start our study of the height of the intersection of the line

$$C = Z(x_0+x_1+x_2)\subset\mathbb{P}^2(\overline{\mathbb{Q}})$$

with its translate ωC by a torsion point ω of $\mathbb{G}_{\mathrm{m}}^2(\overline{\mathbb{Q}})$. As in (1.4), this translate coincides with the zero set of the twist by ω^{-1} of the linear polynomial $x_0+x_1+x_2$, that is

$$\omega C = Z(x_0+\omega_1^{-1}x_1+\omega_2^{-1}x_2).$$

Hence the intersection $C\cap\omega C$ coincides with the solution set of the system of linear equations

$$x_0+x_1+x_2 = x_0+\omega_1^{-1}x_1+\omega_2^{-1}x_2 = 0.$$

When ω in nontrivial, that is when $\omega\neq(1,1)$, it consists of the point

$$P(\omega) = [\omega_2^{-1}-\omega_1^{-1} : 1-\omega_2^{-1} : \omega_1^{-1}-1]\in\mathbb{P}^2(\overline{\mathbb{Q}}). \tag{2.1}$$

Its height depends nontrivially on ω, as the next example shows.

Example 2.1. For $d\geq 2$ let $\zeta\in\mu_d^\circ$ be a primitive d-root of unity, and consider the torsion point $\omega_d=(\zeta,\zeta^2)$. The corresponding intersection point can be written as $P(\omega_d)=[1:-1-\zeta:\zeta]$, and the Galois orbit of the vector of these homogeneous coordinates is

$$\{(1,-1-\zeta^k,\zeta^k)\mid k\in(\mathbb{Z}/d\mathbb{Z})^\times\}.$$

This is a finite set of cardinality $\varphi(d)$, where φ is the Euler totient function.

For every prime p and every $k\in(\mathbb{Z}/k\,\mathbb{Z})^\times$ we have that

$$|\iota_p(\zeta^k)|_p = 1 \quad\text{and}\quad |\iota_p(\zeta^k+1)|_p = |\iota_p(\zeta^k)+1|_p\leq 1$$

because of the ultrametric inequality for the p-adic absolute value. Hence the formula for the height of $P(\omega_d)$ from Proposition 1.2 reduces to its Archimedean contribution, namely

$$\mathrm{h}(P(\omega_d)) = \frac{1}{\varphi(d)}\sum_{k\in(\mathbb{Z}/d\mathbb{Z})^\times}\log\max(1,|\mathrm{e}^{2\pi ik/d}+1|_\infty)$$
$$= \frac{1}{\varphi(d)}\sum_{k\in(\mathbb{Z}/d\mathbb{Z})^\times}\log\max(1,\sqrt{2+2\cos(2\pi k/d)}).$$

It follows that

$$\mathrm{h}(P(\omega_2)) = \mathrm{h}(P(\omega_3)) = 0,\ \mathrm{h}(P(\omega_4)) = \frac{1}{2}\log(2),\ \mathrm{h}(P(\omega_5)) = \frac{1}{4}\log\left(\frac{3+\sqrt{5}}{2}\right),\ \ldots$$

The main result of this section is the following, in which we establish the range of values of the height of the intersection point $P(\omega)$ and determine the extremal cases.

Proposition 2.2. *Let $\omega \in \mathbb{G}_{\mathrm{m}}^2(\overline{\mathbb{Q}})$ be a nontrivial torsion point. Then the corresponding height verifies the inequalities $0 \leq \mathrm{h}(P(\omega)) \leq \log(2)$. The lower bound is attained exactly when*

$$\omega \in \{(1,\zeta),(\zeta,1),(\zeta,\zeta) \mid \zeta \in \mu_\infty \setminus \{1\}\} \cup \{(\zeta,\zeta^2) \mid \zeta \in \mu_3^\circ\},$$

whereas the upper bound is attained exactly when

$$\omega \in \{(-1,\zeta),(\zeta,-1),(\zeta,-\zeta) \mid \zeta \in \mu_\infty \text{ with } \operatorname{ord}(\zeta) \neq 2^k \text{ for all } k \geq 0\}.$$

Before proving the proposition, we choose the vector of homogeneous coordinates of $P(\omega)$ given by

$$\mathcal{P}(\omega) = (\omega_2^{-1} - \omega_1^{-1}, 1 - \omega_2^{-1}, \omega_1^{-1} - 1) \in \overline{\mathbb{Q}}^3. \tag{2.2}$$

The next lemma gives a formula for its local heights.

Lemma 2.3. *Set $d = \operatorname{ord}(\omega)$. Then for each $v \in M_{\mathbb{Q}}$ we have that*

$$\mathrm{h}_v(\mathcal{P}(\omega)) = \frac{1}{\varphi(d)} \sum_{k \in (\mathbb{Z}/d\mathbb{Z})^\times} \log\max(|\iota_v(\omega_2^k) - \iota_v(\omega_1^k)|_v, |\iota_v(\omega_2^k) - 1|_v, |\iota_v(\omega_1^k) - 1|_v),$$

where ι_v is the embedding in (1.7).

Proof. Both ω_1 and ω_2 are contained in the d-th cyclotomic extension of $\mathbb{Q}$, and so are the coordinates of $\mathcal{P}(\omega)$. Hence the Galois orbit of this vector coincides with its orbit under the action of the Galois group of this cyclotomic extension. The latter is isomorphic to $(\mathbb{Z}/d\mathbb{Z})^\times$, and by (1.1) the action of each element k of this group maps ω_i to ω_i^k for $i = 1, 2$. Hence the Galois orbit of $\mathcal{P}(\omega)$ writes down as

$$\begin{aligned} O(\mathcal{P}(\omega)) &= \{(\omega_2^{-k} - \omega_1^{-k}, 1 - \omega_2^{-k}, \omega_1^{-k} - 1) \mid k \in (\mathbb{Z}/d\mathbb{Z})^\times\} \\ &= \{(\omega_2^{k} - \omega_1^{k}, 1 - \omega_2^{k}, \omega_1^{k} - 1) \mid k \in (\mathbb{Z}/d\mathbb{Z})^\times\}. \end{aligned}$$

The elements in this last set are pairwise distinct as k ranges in $(\mathbb{Z}/d\mathbb{Z})^\times$. Indeed any element k in the stabilizer of $\mathcal{P}(\omega)$ has to satisfy $\omega^k = \omega$ and then it must be trivial in $(\mathbb{Z}/d\mathbb{Z})^\times$ as a consequence of the hypothesis that ω has order d. The statement then follows from Proposition 1.2. □

We will need two further auxiliary results. The first is the classical formula for the value of a cyclotomic polynomial at 1, which will also play an important role in Sections 3 and 4. Its proof is elementary and can be found in [Lan94, page 74].

Lemma 2.4. *For $d \geq 2$ let Φ_d be the d-th cyclotomic polynomial. Then*

$$\Phi_d(1) = \begin{cases} p & \text{if } d \text{ is a power of a prime } p, \\ 1 & \text{otherwise.} \end{cases}$$

The second auxiliary result gives the p-adic distance of a root of unity to the point 1.

Lemma 2.5. *Let p be a prime and $d \geq 2$. Then for all $\zeta \in \mu_d^\circ$ we have that*

$$|\iota_p(\zeta) - 1|_p = \begin{cases} p^{-1/\varphi(d)} & \text{if } d \text{ is a power of } p, \\ 1 & \text{otherwise.} \end{cases}$$

Proof. The Galois conjugates of ζ are the elements of the form ζ^k for $k \in (\mathbb{Z}/d\mathbb{Z})^\times$. For each k we have that $\zeta^k - 1 = (\zeta^{k-1} + \cdots + 1)(\zeta - 1)$ and so the ultrametric inequality implies that

$$|\iota_p(\zeta^k) - 1|_p \leq |\iota_p(\zeta) - 1|_p.$$

By symmetry, the reverse inequality also holds. Hence $|\iota_p(\zeta^k) - 1|_p = |\iota_p(\zeta) - 1|_p$ for all k, and so

$$|\iota_p(\zeta) - 1|_p^{\varphi(d)} = \prod_k |\iota_p(\zeta^k) - 1|_p = |\Phi_d(1)|_p.$$

The result follows then from Lemma 2.4. □

Proof of Proposition 2.2. The lower bound comes from the fact that the height is nonnegative. By Kronecker's theorem, the height of $P(\omega)$ vanishes if and only if this point can be written as $[\xi_0 : \xi_1 : \xi_2]$ with $\xi_j \in \mu_\infty \cup \{0\}$ for all j. The determination of such points in C implies that the considered height is equal to 0 if and only if

$$P(\omega) \in \{[1 : -1 : 0], [1 : 0 : -1], [0 : 1 : -1]\} \quad \text{or} \quad P(\omega) \in \{[1 : \zeta : \zeta^2] \mid \zeta \in \mu_3^\circ\}.$$

A comparison with the explicit form of $P(\omega)$ in (2.1) shows that the first alternative holds if and only if ω is equal to $(1, \zeta), (\zeta, 1)$ or (ζ, ζ) with $\zeta \in \mu_\infty \setminus \{1\}$. For the second, we have that $P(\omega) = [1 : \zeta : \zeta^2]$ with $\zeta \in \mu_3^\circ$ if and only if

$$\omega_2^{-1} - \omega_1^{-1} \neq 0, \quad \frac{1 - \omega_2^{-1}}{\omega_2^{-1} - \omega_1^{-1}} = \zeta, \quad \frac{\omega_1^{-1} - 1}{\omega_2^{-1} - \omega_1^{-1}} = \zeta^2. \tag{2.3}$$

This implies that $1 + \zeta\omega_1^{-1} + \zeta^2\omega_2^{-1} = 0$ and so $[1 : \zeta\omega_1^{-1} : \zeta^2\omega_2^{-1}] \in C(\overline{\mathbb{Q}})$. Our previous knowledge of the points of C with homogenous coordinates that are roots of unity implies that either $\omega_1 = \omega_2 = 1$ or $\omega_1 = \zeta^2$, $\omega_2 = \zeta$. Since the second possibility is the only one satisfying the conditions in (2.3), this proves our claim concerning the points of C attaining the lower bound.

Now let $v \in M_\mathbb{Q}$. Using the fact that $|\iota_v(\omega_1)|_v = |\iota_v(\omega_2)|_v = 1$ for every v, the triangular inequality when $v = \infty$ and the ultrametric inequality otherwise, we deduce from Lemma 2.3 that

$$\mathrm{h}_v(\mathcal{P}(\omega)) \leq \begin{cases} \log(2) & \text{if } v = \infty, \\ 0 & \text{if } v \neq \infty, \end{cases} \tag{2.4}$$

which readily implies the upper bound.

This upper bound is attained if and only if all the inequalities in (2.4) are in fact equalities. When $v = \infty$, this requirement forces in particular that the summand for $k = 1$ in the formula in Lemma 2.3 coincides with $\log(2)$, which happens if and only if ω is of the form

$$(-1, \zeta), \quad (\zeta, -1) \quad \text{or} \quad (\zeta, -\zeta) \quad \text{with } \zeta \in \mu_\infty. \tag{2.5}$$

These torsion points have even order and so the indexes in that formula are necessarily odd numbers, which implies that all the summands coincide with $\log(2)$. We conclude that for the Archimedean place the inequality in (2.4) is an equality exactly when ω is of the form described in (2.5).

Set $d = \mathrm{ord}(\zeta)$. To realize the upper bound, for each of the above possibilities for ω we have to furthermore ensure that for every prime p we have that

$$\max(|2|_p, |\iota_p(\zeta^k) - 1|_p, |\iota_p(\zeta^k) + 1|_p) = 1 \quad \text{for all } k \in (\mathbb{Z}/d\mathbb{Z})^\times.$$

This is nontrivial only when $p = 2$, in which case it is equivalent to the condition that $|\iota_2(\zeta^k) - 1|_2 = 1$ for all k, because of the ultrametric property. When $d = 1$ this condition fails, whereas when $d \geq 2$ it holds exactly when d is not a power of 2, by Lemma 2.5. This completes the proof. □

Remark 2.6. In the more general context of Arakelov geometry, the results of [MS19] and [Gua18a, Chapter 5] provide upper bounds for the height of a complete intersection. However, neither of them is sharp in our particular situation.

3. The negligibility of the non-Archimedean heights

We now turn to our main object of study, that is the limit value of the height of the intersection of the line C with its translates by torsion points in a strict sequence. Here we focus on the non-Archimedean contribution to these heights, achieving an explicit expression for it.

We first notice that these non-Archimedean local heights can be nonvanishing. In fact, for any non-Archimedean place of $\mathbb{Q}$ it is easy to construct choices of ω for which the corresponding local height is nontrivial, as the next example shows.

Example 3.1. Let p be a prime and $\zeta \in \mu_p^\circ$. Then by Lemmas 2.3 and 2.4,

$$\mathrm{h}_p(\mathcal{P}(\zeta, 1)) = \frac{1}{\varphi(p)} \sum_{k \in (\mathbb{Z}/p\mathbb{Z})^\times} \log |\iota_p(\zeta^k) - 1|_p = \frac{\log |\Phi_p(1)|_p}{p-1} = -\frac{\log(p)}{p-1}.$$

However, the situation emerging from this example already contains the worst possible behavior of these non-Archimedean local heights, as their explicit computation in the next proposition makes evident.

Proposition 3.2. *Let $\omega \in \mathbb{G}_{\mathrm{m}}^2(\overline{\mathbb{Q}})$ be a nontrivial torsion point and p a prime. Then*

$$\mathrm{h}_p(\mathcal{P}(\omega)) = \begin{cases} -\dfrac{\log(p)}{p^{r-1}(p-1)} & \text{if } \operatorname{ord}(\omega) = p^r \text{ for some } r \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Before proving this proposition, we give an elementary lemma.

Lemma 3.3. *Let $(F, |\cdot|)$ be a field equipped with a non-Archimedean absolute value. Then for all $c \in \mathbb{Z}^2$ and $d \in \mathbb{Z}_{\geq 1}$ such that $\gcd(c_1, c_2, d) = 1$ and all primitive d-root of unity ζ in F we have that*

$$\max(|\zeta^{c_1} - 1|, |\zeta^{c_2} - 1|) = |\zeta - 1|.$$

Proof. Since ζ is a root of unity, its absolute value is equal to 1. It follows from the ultrametric inequality that $|\zeta^e - 1| = |\zeta^{e-1} + \cdots + 1|\,|\zeta - 1| \leq |\zeta - 1|$ for every $e \geq 1$. As the same inequality holds for $e \leq -1$ because of the relation $|\zeta^e - 1| = |\zeta^{-e} - 1|$ and it also holds trivially for $e = 0$, we have that

$$|\zeta^e - 1| \leq |\zeta - 1| \quad \text{for all } e \in \mathbb{Z}. \tag{3.1}$$

Therefore, $\max(|\zeta^{c_1} - 1|, |\zeta^{c_2} - 1|) \leq |\zeta - 1|$.

Else, since $\gcd(c_1, c_2, d) = 1$ there is $b \in \mathbb{Z}^2$ such that $b_1c_1 + b_2c_2 = 1 \pmod{d}$. This implies that $\zeta - 1 = \zeta^{b_2c_2}(\zeta^{b_1c_1} - 1) + (\zeta^{b_2c_2} - 1)$, which by the ultrametric inequality and (3.1) ensures that

$$|\zeta - 1| \leq \max(|\zeta^{b_1c_1} - 1|, |\zeta^{b_2c_2} - 1|) \leq \max(|\zeta^{c_1} - 1|, |\zeta^{c_2} - 1|),$$

completing the proof. □

Proof of Proposition 3.2. Set $d = \operatorname{ord}(\omega)$; since ω is nontrivial, we have that $d \geq 2$. Choose $\zeta \in \mu_d^\circ$. As $\omega \in \mu_d^2$, by (1.1) there is $c \in \mathbb{Z}^2$ with $\gcd(c_1, c_2, d) = 1$ such that

$$\omega = (\zeta^{c_1}, \zeta^{c_2}).$$

The ultrametric inequality together with Lemmas 2.3 and 3.3 then implies that

$$\begin{aligned}\mathrm{h}_p(\mathcal{P}(\omega)) &= \frac{1}{\varphi(d)} \sum_k \log \max(|\iota_p(\zeta^{k\,c_2}) - \iota_p(\zeta^{k\,c_1})|_p, |\iota_p(\zeta^{k\,c_2}) - 1|_p, |\iota_p(\zeta^{k\,c_1}) - 1|_p) \\ &= \frac{1}{\varphi(d)} \sum_k \log |\iota_p(\zeta^k) - 1|_p.\end{aligned}$$

Therefore the statement follows from Lemma 2.5. □

Summing over all finite places, Proposition 3.2 shows that the non-Archimedean contribution to the height of $P(\omega)$ is always nonpositive and that moreover it vanishes precisely when the order of ω has at least two different prime factors.

Corollary 3.4. *For every nontrivial torsion point ω of $\mathbb{G}_{\mathrm{m}}^2(\overline{\mathbb{Q}})$ we have that*

$$\sum_{v \in M_{\mathbb{Q}} \setminus \{\infty\}} \mathrm{h}_v(\mathcal{P}(\omega)) = -\frac{\Lambda(\operatorname{ord}(\omega))}{\varphi(\operatorname{ord}(\omega))},$$

where Λ denotes the von Mangoldt function.

Remark 3.5. More explicitly, Corollary 3.4 says that this part of the height of $P(\omega)$ is equal to

$$-\frac{\log(p)}{p^{r-1}(p-1)}$$

if $\operatorname{ord}(\omega) = p^r$ for some prime p and $r \geq 1$, and to 0 otherwise.

In turn, this result implies that the non-Archimedean contribution to the height of $P(\omega)$ approaches to zero when $\operatorname{ord}(\omega)$ is large.

Corollary 3.6. *Let $(\omega_\ell)_{\ell \geq 1}$ be a sequence of nontrivial torsion points of $\mathbb{G}_{\mathrm{m}}^2(\overline{\mathbb{Q}})$ with $\lim_{\ell \to +\infty} \operatorname{ord}(\omega_\ell) = +\infty$. Then*

$$\lim_{\ell \to +\infty} \sum_{v \in M_{\mathbb{Q}} \setminus \{\infty\}} \mathrm{h}_v(\mathcal{P}(\omega_\ell)) = 0.$$

Proof. By Corollary 3.4 and Remark 3.5, we can reduce without loss of generality to the case when $\operatorname{ord}(\omega_\ell) = p_\ell^{r_\ell}$ with p_ℓ a prime and $r_\ell \geq 1$ for all ℓ. Then the sum of the non-Archimedean local heights of the vector $\mathcal{P}(\omega_\ell)$ is, up to sign, equal to

$$\frac{\log(p_\ell)}{p_\ell^{r_\ell - 1}(p_\ell - 1)} = \frac{p_\ell}{r_\ell\,(p_\ell - 1)} \frac{\log(p_\ell^{r_\ell})}{p_\ell^{r_\ell}} = \frac{p_\ell}{r_\ell\,(p_\ell - 1)} \frac{\log(\operatorname{ord}(\omega_\ell))}{\operatorname{ord}(\omega_\ell)}.$$

Since the first factor in the right-hand side is bounded, this quantity tends to 0 whenever $\ell \to +\infty$. □

4. The limit of the Archimedean height

Having computed the non-Archimedean contribution to the height of the intersection of the line C with its translate by a nontrivial torsion point, we turn to the limit behavior of its Archimedean counterpart for strict sequences of such points.

Set for short $|\cdot| = |\cdot|_\infty$. We denote by

$$\mathbb{S} = (S^1)^2 = \{z \in (\mathbb{C}^\times)^2 \mid |z_1| = |z_2| = 1\}$$

the compact torus of the complex torus $\mathbb{G}_{\mathrm{m}}^2(\mathbb{C}) = (\mathbb{C}^\times)^2$ and by ν its probability Haar measure. Consider the function

$$F\colon (\mathbb{C}^\times)^2 \longrightarrow \mathbb{R} \cup \{-\infty\}, \quad z \longmapsto \log\max(|z_2 - z_1|, |z_2 - 1|, |z_1 - 1|). \tag{4.1}$$

Consider also the *co-tropicalization map*

$$\mathrm{cotrop}\colon (\mathbb{C}^\times)^2 \longrightarrow (\mathbb{R}/2\pi\mathbb{Z})^2, \quad z \longmapsto (\arg(z_1), \arg(z_2))$$

and the function

$$f\colon (\mathbb{R}/2\pi\mathbb{Z})^2 \longrightarrow \mathbb{R} \cup \{-\infty\}, \quad u \longmapsto \log\max(|\mathrm{e}^{\mathrm{i}u_2} - \mathrm{e}^{\mathrm{i}u_1}|, |\mathrm{e}^{\mathrm{i}u_2} - 1|, |\mathrm{e}^{\mathrm{i}u_1} - 1|). \tag{4.2}$$

The direct image measure $\mathrm{cotrop}_*\,\nu$ coincides with the normalized Lebesgue measure on $(\mathbb{R}/2\pi\mathbb{Z})^2$, and the inverse image $\mathrm{cotrop}^*\,f$ coincides on $\mathbb{S}$ with the restriction of F.

The next result is an asymptotic Archimedean counterpart of Corollary 3.6.

Proposition 4.1. *Let $(\omega_\ell)_{\ell\geq 1}$ be a strict sequence in $\mathbb{G}_{\mathrm{m}}^2(\overline{\mathbb{Q}})$ of nontrivial torsion points. Then*

$$\lim_{\ell\to+\infty} \mathrm{h}_\infty(\mathcal{P}(\omega_\ell)) = \int_{\mathbb{S}} F\,d\nu = \frac{1}{(2\pi)^2}\int_{(\mathbb{R}/2\pi\mathbb{Z})^2} f(u)\,du_1du_2.$$

For its proof, we need the next lemma. Given $d, e \geq 1$ with $e \mid d$, consider the associated *reduction homomorphism* between the respective groups of modular units

$$\pi_{d,e}\colon (\mathbb{Z}/d\mathbb{Z})^\times \longrightarrow (\mathbb{Z}/e\mathbb{Z})^\times.$$

Let $d = \prod_p p^{r_p}$ and $e = \prod_p p^{s_p}$ be their respective irreducible factorizations. Under the splitting given by the Chinese reminder theorem we can write

$$\pi_{d,e} = \bigoplus_p \pi_{p^{r_p},p^{s_p}}, \tag{4.3}$$

where $\pi_{p^{r_p},p^{s_p}}\colon (\mathbb{Z}/p^{r_p}\mathbb{Z})^\times \to (\mathbb{Z}/p^{s_p}\mathbb{Z})^\times$ denotes the corresponding reduction map.

Lemma 4.2. *The homomorphism $\pi_{d,e}$ is surjective.*

Proof. The splitting in (4.3) allows to reduce to the case in which $d = p^r$ and $e = p^s$ with $r \geq s \geq 0$. The statement follows then from the fact that an element $k \in \mathbb{Z}$ is a unit modulo p^r if and only if it is a unit modulo p^s, since both conditions are equivalent to $p \nmid k$. □

In the setting of Proposition 4.1, for each $\ell \geq 1$ Lemma 2.3 shows that the Archimedean local height corresponding to the torsion point ω_ℓ writes down as

$$\mathrm{h}_\infty(\mathcal{P}(\omega_\ell)) = \frac{1}{\varphi(d_\ell)}\sum_k F(\iota_\infty(\omega_\ell^k)) \tag{4.4}$$

with $d_\ell = \mathrm{ord}(\omega_\ell)$ and k ranging over the elements of $(\mathbb{Z}/d_\ell\mathbb{Z})^\times$.

Consider the uniform probability measure on the ∞-adic Galois orbit of ω_ℓ, that is the discrete measure on the compact torus defined as

$$\delta_{O(\omega_\ell)_\infty} = \frac{1}{\varphi(d_\ell)} \sum_k \delta_{\iota_\infty(\omega_\ell^k)}, \tag{4.5}$$

where each $\delta_{\iota_\infty(\omega_\ell^k)}$ denotes the Dirac delta measure on the corresponding point. Then the formula in (4.4) can be written as

$$\mathrm{h}_\infty(\mathcal{P}(\omega_\ell)) = \int_{\mathbb{S}} F \, d\delta_{O(\omega_\ell)_\infty}. \tag{4.6}$$

It is well-known that the sequence of probability measures in (4.5) converges weakly to the probability Haar measure ν as $\ell \to +\infty$. Precisely, for any bounded and ν-almost everywhere continuous real-valued function ϕ on $\mathbb{S}$,

$$\lim_{\ell \to +\infty} \int_{\mathbb{S}} \phi \, d\delta_{O(\omega_\ell)_\infty} = \int_{\mathbb{S}} \phi \, d\nu. \tag{4.7}$$

When ϕ is continuous, this is a particular case of Bilu's equidistribution theorem for the Galois orbits of points of small height [Bil97], whereas its extension to the situation when ϕ is just bounded and ν-almost everywhere continuous follows from general results of measure theory like [CT09, Lemme 6.3]. But since the limit in (4.7) concerns torsion points and not arbitrary points of small height, it can also be proven in an elementary way using classical facts on Gaussian exponential sums, as it is done for its quantitative version in [DH19, Proposition 3.3].

In view of (4.6), Proposition 4.1 can be seen as an equidistribution result for a test function ϕ that is continuous everywhere except at the point $z = (1, 1)$, where it has a logarithmic singularity. Hence it is a particular case of both Chambert-Loir and Thuillier's logarithmic equidistribution theorem for Galois orbits of points of small height [CT09], and of Dimitrov and Habegger's logarithmic equidistribution theorem for Galois orbits of torsion points of algebraic tori [DH19]. In spite of that, we give here a simple proof relying solely on the more classical equidistribution theorem in (4.7) and on the formula for the value of a cyclotomic polynomial at 1.

Proof of Proposition 4.1. For each $\ell \geq 1$ set $d_\ell = \mathrm{ord}(\omega_\ell)$ and $e_\ell = \mathrm{ord}(\omega_{\ell,1})$, the latter being a divisor of the first. Since the sequence $(\omega_\ell)_{\ell \geq 1}$ is strict, the condition (1.2) implies that

$$\lim_{\ell \to +\infty} e_\ell = +\infty. \tag{4.8}$$

Consider the function $G\colon (\mathbb{C}^\times)^2 \to \mathbb{R} \cup \{-\infty\}$ defined as $G(z) = \log|z_1 - 1|$. Its integral with respect to the measure ν is the logarithmic Mahler measure of the polynomial $z_1 - 1$, and Jensen's formula applied to it shows that this quantity vanishes. On the other hand, its integral with respect to the discrete measure in (4.5) can be computed as

$$\begin{aligned}\int_{\mathbb{S}} G \, d\delta_{O(\omega_\ell)_\infty} &= \frac{1}{\varphi(d_\ell)} \sum_{k \in (\mathbb{Z}/d_\ell\mathbb{Z})^\times} \log|\iota_\infty(\omega_{\ell,1})^k - 1| \\ &= \frac{1}{\varphi(e_\ell)} \sum_{k \in (\mathbb{Z}/e_\ell\mathbb{Z})^\times} \log|\iota_\infty(\omega_{\ell,1})^k - 1| = \frac{1}{\varphi(e_\ell)} \log|\Phi_{e_\ell}(1)|.\end{aligned}$$

The second equality follows from the fact that the summand indexed by an element $k \in (\mathbb{Z}/d_\ell\mathbb{Z})^\times$ takes a value that depends only on its image under the reduction homomorphism π_{d_ℓ,e_ℓ}, which by Lemma 4.2 is surjective with fibers of cardinality $\varphi(d_\ell)/\varphi(e_\ell)$. Hence Lemma 2.4 together with (4.8) and the same argument in the proof of Corollary 3.6 implies that

$$\lim_{\ell\to+\infty}\int_{\mathbb{S}} G\,d\delta_{O(\omega_\ell)_\infty} = 0 = \int_{\mathbb{S}} G\,d\nu. \tag{4.9}$$

Now let $(U_m)_{m\geq 1}$ be the nested sequence of neighborhoods of the closed subset $\{z\in\mathbb{S}\mid z_1=1\}$ of the compact torus defined as

$$U_m = \Big\{z\in\mathbb{S}\mid \arg(z_1)\in\Big(-\frac{1}{m},\frac{1}{m}\Big)\pmod{2\pi}\Big\}.$$

Since both F and G are continuous outside that closed subset, for each $m\geq 1$ the equidistribution theorem in (4.7) shows that

$$\lim_{\ell\to+\infty}\int_{\mathbb{S}\setminus U_m} F\,d\delta_{O(\omega_\ell)_\infty} = \int_{\mathbb{S}\setminus U_m} F\,d\nu,\quad \lim_{\ell\to+\infty}\int_{\mathbb{S}\setminus U_m} G\,d\delta_{O(\omega_\ell)_\infty} = \int_{\mathbb{S}\setminus U_m} G\,d\nu. \tag{4.10}$$

The second limit in (4.10) together with that in (4.9) implies that

$$\lim_{\ell\to+\infty}\int_{U_m} G\,d\delta_{O(\omega_\ell)_\infty} = -\lim_{\ell\to+\infty}\int_{\mathbb{S}\setminus U_m} G\,d\delta_{O(\omega_\ell)_\infty} = -\int_{\mathbb{S}\setminus U_m} G\,d\nu = \int_{U_m} G\,d\nu. \tag{4.11}$$

We have that $G(z)\leq F(z)\leq \log(2)$ for all $z\in\mathbb{S}$ and so for each $\ell\geq 1$,

$$\int_{U_m} G\,d\delta_{O(\omega_\ell)_\infty} \leq \int_{U_m} F\,d\delta_{O(\omega_\ell)_\infty} \leq \int_{U_m} \log(2)\,d\delta_{O(\omega_\ell)_\infty}. \tag{4.12}$$

We deduce from (4.10), (4.11) and (4.12) that

$$\begin{aligned}\int_{\mathbb{S}\setminus U_m} F\,d\nu + \int_{U_m} G\,d\nu &\leq \liminf_{\ell}\int_{\mathbb{S}} F\,d\delta_{O(\omega_\ell)_\infty}\\ &\leq \limsup_{\ell}\int_{\mathbb{S}} F\,d\delta_{O(\omega_\ell)_\infty} \leq \int_{\mathbb{S}\setminus U_m} F\,d\nu + \log(2)\,\nu(U_m).\end{aligned}$$

The first equality in the statement follows taking the limit for $m\to+\infty$, using the fact that $\lim_{m\to+\infty}\nu(U_m)=0$ and the absolute continuity of the Lebesgue integral, whereas the second is a direct consequence of the first together with the change of variables formula. □

5. Computing the integral

Next we compute the integral giving the limit of the Archimedean heights of the vectors $\mathcal{P}(\omega_\ell)$ for a strict sequence of nontrivial torsion points (Proposition 4.1), namely

$$I = \frac{1}{(2\pi)^2}\int_{(\mathbb{R}/2\pi\mathbb{Z})^2} \log\max(|\mathrm{e}^{\mathrm{i}u_2}-\mathrm{e}^{\mathrm{i}u_1}|, |\mathrm{e}^{\mathrm{i}u_2}-1|, |\mathrm{e}^{\mathrm{i}u_1}-1|)\,du_1du_2.$$

More precisely, we will prove the following.

Proposition 5.1. $I = \dfrac{2\,\zeta(3)}{3\,\zeta(2)} = 0.487175\ldots$.

We will exploit the existing symmetries to simplify the calculation of this integral. To this end, consider the linear automorphisms of $\mathbb{R}^2$ given as

$$\alpha(u_1,u_2)=(u_2,u_1),\quad \beta(u_1,u_2)=(-u_2,u_1-u_2),\quad \gamma(u_1,u_2)=(-u_1,-u_2). \tag{5.1}$$

Since they restrict to automorphisms of the lattice $(2\pi\mathbb{Z})^2$, each of them also induces an automorphism of the quotient space $(\mathbb{R}/2\pi\mathbb{Z})^2$, that we denote with the corresponding overlined letter. We set H for the group of automorphisms of $(\mathbb{R}/2\pi\mathbb{Z})^2$ generated by them.

Recall that a *fundamental domain* for the action of H on $(\mathbb{R}/2\pi\mathbb{Z})^2$ is a closed subset of $(\mathbb{R}/2\pi\mathbb{Z})^2$ whose translates by elements of H cover this space, and such that the intersection of any two different translates has empty interior.

For the sequel, denote by D the triangle of $\mathbb{R}^2$ with vertices $(0,0)$, $(\pi,0)$ and $(4\pi/3,2\pi/3)$, and by $\overline{D}$ its image in $(\mathbb{R}/2\pi\mathbb{Z})^2$. Set also f for the integrand of I, which coincides with the function in (4.2).

Proposition 5.2. *The automorphism group H verifies the following properties:*
(1) $H=\{\bar\alpha^{e_1}\,\bar\beta^{e_2}\,\bar\gamma^{e_3}\mid e_1,e_3=0,1,\ e_2=0,1,2\}$,
(2) $\#H=12$,
(3) the set $\overline{D}$ is a fundamental domain for the action of H,
(4) the Lebesgue measure on $(\mathbb{R}/2\pi\mathbb{Z})^2$ is invariant under H,
(5) the function f is invariant under H,
(6) for each $u\in\overline{D}$ we have that $f(u)=\log|\mathrm{e}^{\mathrm{i}u_1}-1|$.

Proof. The generators of H verify the relations

$$\bar\alpha^2=\bar\beta^3=\bar\gamma^2=1,\quad \bar\gamma\,\bar\alpha=\bar\alpha\,\bar\gamma,\quad \bar\gamma\,\bar\beta=\bar\beta\,\bar\gamma,\quad \bar\beta\,\bar\alpha=\bar\alpha\,\bar\beta^2.$$

The last three imply that all the elements of H are of the form $\bar\alpha^{e_1}\,\bar\beta^{e_2}\,\bar\gamma^{e_3}$ with $e_1,e_2,e_3\geq 0$, whereas the others give the stated upper bounds for these exponents, proving (1).

To prove (3), notice that the action of β on the nonzero vertices of D is given by

$$(\pi,0)\xrightarrow{\beta}(0,\pi)\xrightarrow{\beta}(-\pi,-\pi),\qquad \left(\tfrac{4\pi}{3},\tfrac{2\pi}{3}\right)\xrightarrow{\beta}\left(\tfrac{-2\pi}{3},\tfrac{2\pi}{3}\right)\xrightarrow{\beta}\left(\tfrac{-2\pi}{3},\tfrac{-4\pi}{3}\right).$$

The actions on $\mathbb{R}^2$ of α and γ are both easy to visualize, since these linear maps are the symmetries with respect to the diagonal line and to the origin, respectively. The first picture in Figure 5.1 describes the action on D of these three linear maps and their compositions, whereas the second shows these regions on the quotient space $(\mathbb{R}/2\pi\mathbb{Z})^2$, represented by the square $[0,2\pi]^2$ with opposite edges identified.

This second picture shows that the only product $\bar\alpha^{e_1}\,\bar\beta^{e_2}\,\bar\gamma^{e_3}$ with $e_1,e_3=0,1$ and $e_2=0,1,2$ that fixes $\overline{D}$ occurs when $e_1=e_2=e_3=0$. This proves (2). The statement in (3) can also be checked from the picture: the translates of $\overline{D}$ by the elements of H fit in $(\mathbb{R}/2\pi\mathbb{Z})^2$ like the pieces of a puzzle. Hence these translates cover the whole of the space, and the intersections of any two different translates has empty interior.

To prove the statement in (4), note that the linear automorphisms in (5.1) preserve the Lebesgue measure of $\mathbb{R}^2$ because their determinants are equal to 1, and so do the induced automorphisms of $(\mathbb{R}/2\pi\mathbb{Z})^2$.

For (5), consider the functions on $(\mathbb{R}/2\pi\mathbb{Z})^2$ defined as

$$\phi_1(u)=|\mathrm{e}^{\mathrm{i}u_2}-\mathrm{e}^{\mathrm{i}u_1}|,\quad \phi_2(u)=|\mathrm{e}^{\mathrm{i}u_1}-1|,\quad \phi_3(u)=|\mathrm{e}^{\mathrm{i}u_2}-1|.$$

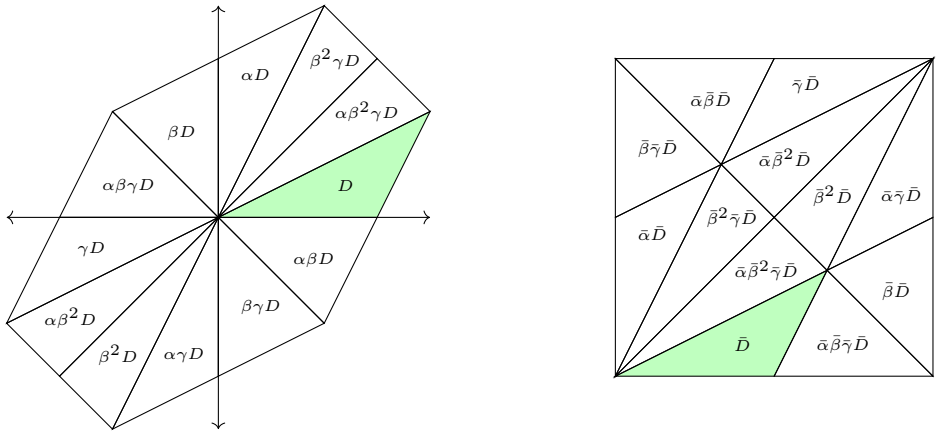

FIGURE 5.1. The action of H on the fundamental domain $\overline{D}$

They are invariant under $\bar\gamma$, whereas $\bar\alpha$ leaves invariant ϕ_1 and exchanges ϕ_2 with ϕ_3, and β makes a cyclic permutation. We have that $f = \log\max(\phi_1, \phi_2, \phi_3)$ and so this function is invariant under H.

Finally, note that $|\mathrm{e}^{\mathrm{i}s} - 1| = \sqrt{2 - 2\cos(s)}$ and so for $s, s' \in [0, 2\pi]$,

$$|\mathrm{e}^{\mathrm{i}s} - 1| \le |\mathrm{e}^{\mathrm{i}s'} - 1| \text{ if and only if } s \le s' \le 2\pi - s. \tag{5.2}$$

On the other hand, a point $u \in \mathbb{R}^2$ lies in D if and only if it verifies the inequalities

$$u_2 \ge 0, \quad u_1 - 2u_2 \ge 0, \quad u_2 - 2u_1 + 2\pi \ge 0.$$

In particular, $u_1 - u_2 \in [0, 2\pi]$ for every $u \in D$. These inequalities imply those in (5.2) for $s' = u_1$ and $s = u_1 - u_2, u_2$. Hence $\phi_2 \ge \phi_1, \phi_3$ on $\overline{D}$, which proves (6). □

We also need the next integral formulæ.

Lemma 5.3. *The following equalities hold:*

(1) $\displaystyle\int_0^{\pi} s \log|\mathrm{e}^{\mathrm{i}s} - 1|\, ds = \frac{7}{4}\,\zeta(3),$

(2) $\displaystyle\int_{\pi}^{\frac{4\pi}{3}} (4\pi - 3s) \log|\mathrm{e}^{\mathrm{i}s} - 1|\, ds = \frac{11}{12}\,\zeta(3).$

Proof. Let $0 < \varepsilon \le \pi$. The Dirichlet-Hardy test for the convergence of series [JJ56, page 42] implies that the sequence of partial sums $\sum_{k=1}^{\ell} \mathrm{e}^{\mathrm{i}ks}/k$, $\ell \ge 1$, converges uniformly on the interval $[\varepsilon, \pi]$ to the principal determination of $-\log(1 - \mathrm{e}^{\mathrm{i}s})$. Hence the sequence

$$\mathrm{Re}\left(\sum_{k=1}^{\ell} \frac{\mathrm{e}^{\mathrm{i}ks}}{k}\right), \quad \ell \ge 1, \tag{5.3}$$

converges uniformly on this interval to the function $-\log|\mathrm{e}^{\mathrm{i}s} - 1|$. This implies that

$$\int_{\varepsilon}^{\pi} s \log|\mathrm{e}^{\mathrm{i}s} - 1|\, ds = -\sum_{k\ge1} \frac{1}{k}\,\mathrm{Re}\left(\int_{\varepsilon}^{\pi} s\,\mathrm{e}^{\mathrm{i}ks}\, ds\right). \tag{5.4}$$

For each $k \ge 1$, integration by parts gives

$$\int_{\varepsilon}^{\pi} s\,\mathrm{e}^{\mathrm{i}ks}\, ds = \frac{1}{\mathrm{i}k}\left(\Big[s\,\mathrm{e}^{\mathrm{i}ks}\Big]_{\varepsilon}^{\pi} - \int_{\varepsilon}^{\pi} \mathrm{e}^{\mathrm{i}ks}\, ds\right) = \frac{1}{\mathrm{i}k}\left(\pi(-1)^k - \varepsilon\,\mathrm{e}^{\mathrm{i}k\varepsilon} - \frac{1}{\mathrm{i}k}((-1)^k - \mathrm{e}^{\mathrm{i}k\varepsilon})\right).$$

We deduce from (5.4) that

$$\int_\varepsilon^\pi s\log|\mathrm{e}^{\mathrm{i}s}-1|\,ds=\sum_k\Big(\frac{\varepsilon}{k^2}\operatorname{Im}(\mathrm{e}^{\mathrm{i}k\varepsilon})+\frac{1}{k^3}(\operatorname{Re}(\mathrm{e}^{\mathrm{i}k\varepsilon})-(-1)^k)\Big).$$

Taking the limit $\varepsilon\to 0$ we obtain that

$$\begin{aligned}\int_0^\pi s\log|\mathrm{e}^{\mathrm{i}s}-1|\,ds&=\sum_k\frac{1-(-1)^k}{k^3}=2\sum_{2\nmid k}\frac{1}{k^3}\\&=2\Big(\sum_k\frac{1}{k^3}-\sum_{2|k}\frac{1}{k^3}\Big)=2\Big(1-\frac{1}{8}\Big)\sum_k\frac{1}{k^3}=\frac{7}{4}\zeta(3),\end{aligned}$$

proving (1). For the formula in (2), we deduce from the uniform convergence of the sequence in (5.3) that

$$\int_\pi^{\frac{4\pi}{3}}(4\pi-3s)\log|\mathrm{e}^{\mathrm{i}s}-1|\,ds=-\sum_k\frac{1}{k}\operatorname{Re}\Big(\int_\pi^{\frac{4\pi}{3}}(4\pi-3s)\,\mathrm{e}^{\mathrm{i}ks}\,ds\Big).\tag{5.5}$$

For each $k\geq 1$, integrating by parts now gives

$$\begin{aligned}\int_\pi^{\frac{4\pi}{3}}(4\pi-3s)\,\mathrm{e}^{\mathrm{i}ks}\,ds&=\frac{1}{\mathrm{i}k}\Big(\Big[(4\pi-3s)\,\mathrm{e}^{\mathrm{i}ks}\Big]_\pi^{\frac{4\pi}{3}}+3\int_\pi^{\frac{4\pi}{3}}\mathrm{e}^{\mathrm{i}ks}\,ds\Big)\\&=\frac{1}{\mathrm{i}k}\Big(-\pi\,(-1)^k+\frac{3}{\mathrm{i}k}\,(\rho^k-(-1)^k)\Big)\end{aligned}$$

with $\rho=\mathrm{e}^{\mathrm{i}\frac{4\pi}{3}}=\frac{-1}{2}-\mathrm{i}\,\frac{\sqrt{3}}{2}$. By (5.5), we conclude that

$$\begin{aligned}\int_\pi^{\frac{4\pi}{3}}(4\pi-3s)\log|\mathrm{e}^{\mathrm{i}s}-1|\,ds&=3\sum_k\frac{1}{k^3}(\operatorname{Re}(\rho^k)-(-1)^k)\\&=3\Big(\sum_{3|k}\frac{1}{k^3}-\frac{1}{2}\sum_{3\nmid k}\frac{1}{k^3}-\sum_{2|k}\frac{1}{k^3}+\sum_{2\nmid k}\frac{1}{k^3}\Big)\\&=3\Big(\frac{3}{2}\sum_{3|k}\frac{1}{k^3}-\frac{1}{2}\sum_k\frac{1}{k^3}-2\sum_{2|k}\frac{1}{k^3}+\sum_k\frac{1}{k^3}\Big)\\&=3\Big(\frac{3}{2}\,\frac{1}{27}-\frac{1}{2}-2\,\frac{1}{8}+1\Big)\sum_k\frac{1}{k^3}\\&=\frac{11}{12}\,\zeta(3),\end{aligned}$$

as stated. □

Proof of Proposition 5.1. Proposition 5.2 together with Fubini's theorem implies that

$$I=\frac{12}{(2\pi)^2}\int_{\overline{D}}\log|\mathrm{e}^{\mathrm{i}u_1}-1|\,du_1du_2=\frac{12}{(2\pi)^2}\int_0^{\frac{4\pi}{3}}\min\Big(\frac{u_1}{2},2\pi-\frac{3u_1}{2}\Big)\log|\mathrm{e}^{\mathrm{i}u_1}-1|\,du_1.$$

By Lemma 5.3, this quantity is equal to

$$\frac{12}{(2\pi)^2}\Big(\frac{1}{2}\,\frac{7}{4}\,\zeta(3)+\frac{1}{2}\,\frac{11}{12}\,\zeta(3)\Big)=\frac{4\,\zeta(3)}{\pi^2}=\frac{2\,\zeta(3)}{3\,\zeta(2)}$$

thanks to Euler's formula $\zeta(2)=\pi^2/6$. □

6. The distribution of the height

Finally we can join together the different pieces from the previous sections to obtain our first main result. Set for short

$$\eta = \frac{2\,\zeta(3)}{3\,\zeta(2)} = 0.487175\ldots.$$

Theorem 6.1. *Let $(\omega_\ell)_{\ell\geq 1}$ be a strict sequence in $\mathbb{G}_{\mathrm{m}}^2(\overline{\mathbb{Q}})$ of nontrivial torsion points. Then*

$$\lim_{\ell\to+\infty} \mathrm{h}(C\cap\omega_\ell C) = \eta.$$

Proof. This follows readily from the definition of the height in (1.6) together with Corollary 3.6 and Propositions 4.1 and 5.1. □

The following questions are natural in this context.

Question 6.2. *Is it possible to obtain a quantitative version of Theorem 6.1? This is understood as an estimate, for a given nontrivial torsion point ω, of the discrepancy between the height of $C\cap\omega C$ and the limit value η in terms of the strictness degree of ω, that is, the minimal degree of a 1-dimensional algebraic subgroup containing it.*

Question 6.3. *Can Theorem 6.1 be extended to points of small height? It would be interesting to prove (or disprove) that its conclusion holds for a strict sequence in $\mathbb{G}_{\mathrm{m}}^2(\overline{\mathbb{Q}})$ of points whose height converges to zero, but are not necessarily torsion.*

The notion of strict sequence of points can be extended to include the finite subsets that appear naturally when doing statistics on the values of the height, as follows.

Definition 6.4. Let V be an algebraic subset of $\mathbb{G}_{\mathrm{m}}^n(\overline{\mathbb{Q}})$. A sequence $(E_\ell)_{\ell\geq 1}$ of nonempty finite subsets of V is *strict* if for every algebraic subgroup $H\subset\mathbb{G}_{\mathrm{m}}^n(\overline{\mathbb{Q}})$ not containing V we have that

$$\lim_{\ell\to+\infty}\frac{\#(E_\ell\cap H)}{\#E_\ell} = 0.$$

The situation of interest in this section is the case $V=\mathbb{G}_{\mathrm{m}}^n(\overline{\mathbb{Q}})$. The criterion in (1.2) can be extended to this setting: a sequence $(E_\ell)_{\ell\geq 1}$ is strict in $\mathbb{G}_{\mathrm{m}}^n(\overline{\mathbb{Q}})$ if and only if we have that

$$\lim_{\ell\to+\infty}\frac{\#(E_\ell\cap\mathrm{Ker}(\chi^a))}{\#E_\ell} = 0 \quad \text{for all } a\in\mathbb{Z}^n\setminus\{(0,\ldots,0)\}. \tag{6.1}$$

Example 6.5. The sequence of the sets of d-torsion points μ_d^n of $\mathbb{G}_{\mathrm{m}}^n(\overline{\mathbb{Q}})$ is strict. Indeed, for each $a\in\mathbb{Z}^n\setminus\{(0,\ldots,0)\}$ and $d\geq 1$ we have that

$$\frac{\#(\mu_d^n\cap\mathrm{Ker}(\chi^a))}{\#\mu_d^n} = \frac{\#\,\mathrm{Ker}(\chi^a|_{\mu_d^n})}{\#\mu_d^n} = \frac{1}{\#\,\mathrm{Im}(\chi^a|_{\mu_d^n})} = \frac{\gcd(a_1,\ldots,a_n,d)}{d}, \tag{6.2}$$

where the last equality comes from the fact that the image of μ_d^n under χ^a is generated by $\zeta^{\gcd(a_1,\ldots,a_n)}$ for any primitive d-root of unity ζ. Hence the quotient in (6.2) tends to 0 when $d\to+\infty$. Since this holds for every a, the criterion in (6.1) implies that the sequence $(\mu_d^n)_{d\geq 1}$ is strict.

We next extend Theorem 6.1 to compute the typical value of the height of the intersection of the line C with its translates by the torsion points in a strict sequence of nonempty finite subsets.

Theorem 6.6. *Let $(W_\ell)_{\ell\geq 1}$ be a strict sequence of nonempty finite subsets of $\mathbb{G}_{\mathrm{m}}^2(\overline{\mathbb{Q}})$ of nontrivial torsion points. Then for each $\varepsilon > 0$ we have that*

$$\lim_{\ell\to+\infty} \frac{\#\{\omega \in W_\ell \mid |\,\mathrm{h}(C\cap\omega C) - \eta| < \varepsilon\}}{\#W_\ell} = 1.$$

Moreover $\displaystyle\lim_{\ell\to+\infty} \frac{1}{\#W_\ell} \sum_{\omega\in W_\ell} \mathrm{h}(C\cap\omega C) = \eta.$

This is a consequence of the previous results and the following general transfer principle.

Lemma 6.7. *Let V be an algebraic subset of $\mathbb{G}_{\mathrm{m}}^n(\overline{\mathbb{Q}})$. Let $(E_\ell)_{\ell\geq 1}$ be a strict sequence of nonempty finite subsets of V, ϕ a real-valued function on $\bigcup_{\ell\geq 1} E_\ell$ and κ a real number such that for every strict sequence $(\gamma_\ell)_{\ell\geq 1}$ in V contained in this union we have that $\lim_{\ell\to+\infty}\phi(\gamma_\ell) = \kappa$. Then*

(1) for each $\varepsilon > 0$ we have that $\displaystyle\lim_{\ell\to+\infty} \frac{\#\{\gamma \in E_\ell \mid |\phi(\gamma) - \kappa| < \varepsilon\}}{\#E_\ell} = 1,$

(2) if ϕ is bounded on $\bigcup_{\ell\geq 1} E_\ell$, then $\displaystyle\lim_{\ell\to+\infty} \frac{1}{\#E_\ell} \sum_{\gamma\in E_\ell} \phi(\gamma) = \kappa.$

Proof. We argue by contradiction, supposing that there is $\varepsilon > 0$ for which the statement in (1) does not hold. Restricting to a subsequence, we can then assume that there is $c > 0$ such that

$$\frac{\#\{\gamma \in E_\ell \mid |\phi(\gamma) - \kappa| \geq \varepsilon\}}{\#E_\ell} \geq c \quad \text{for all } \ell \geq 1.$$

Let $(H_j)_{j\geq 1}$ be the complete list of algebraic subgroups of $\mathbb{G}_{\mathrm{m}}^n(\overline{\mathbb{Q}})$ not containing V. Unless $V = \{(1,\dots,1)\}$, in which case the statement is trivial, this is a countably infinite list. Since $(E_\ell)_{\ell\geq 1}$ is strict in V, we can take a subsequence $(E_{\ell_k})_{k\geq 1}$ such that

$$\frac{\#\big(E_{\ell_k} \cap \bigcup_{j=1}^k H_j\big)}{\#E_{\ell_k}} < c \quad \text{for all } k \geq 1.$$

Then for each $k \geq 1$ we can choose $\gamma_k \in E_{\ell_k} \setminus \bigcup_{j=1}^k H_j$ such that

$$|\phi(\gamma_k) - \kappa| \geq \varepsilon. \tag{6.3}$$

The sequence $(\gamma_k)_{k\geq 1}$ is strict in V by construction, and so the inequality (6.3) contradicts the hypothesis and proves (1). The statement in (2) follows easily from (1) and the hypothesis that ϕ is bounded. □

Proof of Theorem 6.6. This is a direct consequence of Lemma 6.7 applied to the strict sequence $(W_\ell)_{\ell\geq 1}$ of $\mathbb{G}_{\mathrm{m}}^2(\overline{\mathbb{Q}})$ and to the real-valued function

$$\bigcup_{\ell\geq 1} W_\ell \longrightarrow \mathbb{R}, \quad \omega \longmapsto \mathrm{h}(C\cap\omega C)$$

together with Theorem 6.1 and Proposition 2.2. □

The next result is an easy consequence of Example 6.5 and Theorem 6.6.

Corollary 6.8. *For each $\varepsilon > 0$ we have that*

$$\lim_{d\to+\infty} \frac{1}{d^2-1} \#\{\omega \in \mu_d^2 \setminus \{(1,1)\} \mid |\mathrm{h}(C \cap \omega C) - \eta| < \varepsilon\} = 1.$$

Moreover $\displaystyle\lim_{d\to+\infty} \frac{1}{d^2-1} \sum_{\omega\in\mu_d^2\setminus\{(1,1)\}} \mathrm{h}(C \cap \omega C) = \eta.$

7. Intermezzo: sequences of torsion points in algebraic subgroups

Here we extend our previous study to sequences of torsion points which are strict in a fixed irreducible component over the rationals of a 1-dimensional algebraic subgroup of the torus, thus finding other interesting limit values for the height.

We start by recalling that the 1-dimensional algebraic subgroups of $\mathbb{G}_{\mathrm{m}}^2(\overline{\mathbb{Q}})$ are the algebraic subsets defined by a binomial of the form $\chi^c - 1$ with $c \in \mathbb{Z}^2 \setminus \{(0,0)\}$. Writing $c = la$ for a primitive vector $a \in \mathbb{Z}^2$ and a positive integer l, its irreducible decomposition over $\mathbb{Q}$ is

$$Z(\chi^c - 1) = \bigcup_{e|l} Z(\Phi_e(\chi^a))$$

with Φ_e the e-th cyclotomic polynomial.

Hence for the rest of this section we fix a primitive vector $a \in \mathbb{Z}^2$ and a positive integer e, and we consider the algebraic subset $V_{a,e} = Z(\Phi_e(\chi^a))$. Its irreducible decomposition over $\overline{\mathbb{Q}}$ is given by the disjoint union

$$V_{a,e} = \bigcup_{\zeta\in\mu_e^\circ} Z(\chi^a - \zeta).$$

Each irreducible component $Z(\chi^a - \zeta)$ is a *torsion curve*, since it is the translate of the 1-dimensional (algebraic) subtorus $Z(\chi^a - 1) \simeq \mathbb{G}_{\mathrm{m}}(\overline{\mathbb{Q}})$ by any torsion point of it.

Let $(\omega_\ell)_{\ell\geq 1}$ be a strict sequence in $V_{a,e}$ of torsion points. Since any algebraic subgroup of the torus not containing $V_{a,e}$ intersects this algebraic subset in a finite number of points, the strictness of $(\omega_\ell)_{\ell\geq 1}$ in $V_{a,e}$ is equivalent to the fact that each torsion point appears at most a finite number of times in the sequence. In turn, this is equivalent to

$$\lim_{\ell\to+\infty} \mathrm{ord}(\omega_\ell) = +\infty. \tag{7.1}$$

Recall that for each ℓ we denote by $\delta_{O(\omega_\ell)_\infty}$ the uniform probability measure on the ∞-adic Galois orbit of ω_ℓ as in (4.5). Our first goal is to determine the limit of this sequence of discrete measures with respect to the weak-$*$ topology. To this end, consider the subset of the compact torus $\mathbb{S}$ defined as

$$\mathbb{S}_{a,e} = \{z \in \mathbb{S} \mid \Phi_e(\chi^a(z)) = 0\} = \bigcup_{\zeta\in\mu_e^\circ} \{z \in \mathbb{S} \mid \chi^a(z) = \iota_\infty(\zeta)\}.$$

It is a disjoint union of translates of the circle $\{z \in \mathbb{S} \mid \chi^a(z) = 1\} \simeq S^1$ containing the ∞-adic Galois orbit of any torsion point in $V_{a,e}$. For each primitive e-root of unity ζ we denote by $\lambda_{a,e,\zeta}$ the probability measure on $\mathbb{S}_{a,e}$ supported on the corresponding translate of S^1, where it coincides with the measure induced by the probability Haar measure of this circle. Consider then the probability measure on $\mathbb{S}_{a,e}$ defined as

$$\nu_{a,e} = \frac{1}{\varphi(e)} \sum_{\zeta\in\mu_e^\circ} \lambda_{a,e,\zeta}.$$

The next result is the analogue in our present situation of (4.7). Its proof is done by reducing to a result of Dimitrov and Habegger in [DH19] on the equidistribution of the orbits of roots of unity under the action of large subgroups of their Galois groups.

Proposition 7.1. *Let $(\omega_\ell)_{\ell\geq 1}$ be a strict sequence in $V_{a,e}$ of torsion points and ϕ a bounded and $\nu_{a,e}$-almost everywhere continuous real-valued function on $\mathbb{S}_{a,e}$. Then*

$$\lim_{\ell\to+\infty}\int_{\mathbb{S}_{a,e}}\phi\,d\delta_{O(\omega_\ell)_\infty}=\int_{\mathbb{S}_{a,e}}\phi\,d\nu_{a,e}. \tag{7.2}$$

Proof. As a is primitive, after a change of variables we can suppose without loss of generality that $a=(0,1)$, so that

$$V_{a,e}=Z(\Phi_e(t_2))=\bigcup_{\zeta\in\mu_e^\circ}\overline{\mathbb{Q}}^\times\times\{\zeta\}\quad\text{and}\quad\mathbb{S}_{a,e}=\bigcup_{\zeta\in\mu_e^\circ}S^1\times\{\iota_\infty(\zeta)\}.$$

For each $\ell\geq 1$ set $d_\ell=\operatorname{ord}(\omega_\ell)$ and $e_\ell=\operatorname{ord}(\omega_{\ell,1})$. Since both e_ℓ and e divide d_ℓ we can consider the reduction homomorphisms

$$\pi_{d_\ell,e_\ell}\colon(\mathbb{Z}/d_\ell\mathbb{Z})^\times\longrightarrow(\mathbb{Z}/e_\ell\mathbb{Z})^\times\quad\text{and}\quad\pi_{d_\ell,e}\colon(\mathbb{Z}/d_\ell\mathbb{Z})^\times\longrightarrow(\mathbb{Z}/e\mathbb{Z})^\times.$$

Set $J_\ell\subset(\mathbb{Z}/d_\ell\mathbb{Z})^\times$ for the kernel of $\pi_{d_\ell,e}$ and $K_\ell\subset(\mathbb{Z}/e_\ell\mathbb{Z})^\times$ for its image under π_{d_ℓ,e_ℓ}. As $\operatorname{ord}(\omega_{\ell,2})=e$, we have that $d_\ell=\operatorname{lcm}(e_\ell,e)$ and so $J_\ell\simeq K_\ell$.

Because of [CT09, Lemme 6.3] we can assume without loss of generality that ϕ is continuous. Then we write each integral in its left hand side of (7.2) as a sum of integrals over the different connected components of $\mathbb{S}_{a,e}$ by suitably partitioning the corresponding Galois orbit $O(\omega_\ell)=\{\omega_\ell^k\mid k\in(\mathbb{Z}/d_\ell\mathbb{Z})^\times\}$. To this aim, for each $\zeta\in\mu_e^\circ$ we choose $r_{\ell,\zeta}\in(\mathbb{Z}/d_\ell\mathbb{Z})^\times$ such that $\omega_{\ell,2}^{r_{\ell,\zeta}}=\zeta$, which is possible because $\pi_{d_\ell,e}$ is surjective thanks to Lemma 4.2. In fact, $\omega_{\ell,2}^k=\zeta$ if and only if $k\in J_\ell\cdot r_{\ell,\zeta}$ and so

$$\begin{aligned}\int_{\mathbb{S}_{a,e}}\phi\,d\delta_{O(\omega_\ell)_\infty}&=\frac{1}{\varphi(d_\ell)}\sum_{k\in(\mathbb{Z}/d_\ell\mathbb{Z})^\times}\phi(\iota_\infty(\omega_\ell^k))\\&=\frac{1}{\varphi(e)}\sum_{\zeta\in\mu_e^\circ}\frac{1}{\#J_\ell}\sum_{j\in J_\ell}\phi\big(\iota_\infty(\omega_{\ell,1}^{j\cdot r_{\ell,\zeta}}),\iota_\infty(\zeta)\big)=\frac{1}{\varphi(e)}\sum_{\zeta\in\mu_e^\circ}\int_{S^1}\phi_\zeta\,d\delta_{\iota_\infty(K_\ell\cdot\omega_{\ell,1}^{r_{\ell,\zeta}})},\end{aligned} \tag{7.3}$$

where ϕ_ζ denotes the continuous function on S^1 defined by $\phi_\zeta(z)=\phi(z,\iota_\infty(\zeta))$, which is integrated against the uniform probability measure on the ∞-adic orbit of $\omega_{\ell,1}^{r_{\ell,\zeta}}$ under the action of K_ℓ.

Since $d_\ell=\operatorname{lcm}(e_\ell,e)$, the condition in (7.1) implies that

$$\lim_{\ell\to+\infty}\operatorname{ord}(\omega_{\ell,1}^{r_{\ell,\zeta}})=\lim_{\ell\to+\infty}e_\ell=+\infty,$$

and so $(\omega_{\ell,1}^{r_{\ell,\zeta}})_{\ell\geq1}$ is a strict sequence in $\mathbb{G}_{\mathrm{m}}(\overline{\mathbb{Q}})$. In the terminology of [DH19, Section 3], the conductor of the subgroup K_ℓ of $(\mathbb{Z}/e_\ell\mathbb{Z})^\times$ is equal to $\gcd(e_\ell,e)$, because this subgroup coincides with the kernel of the reduction homomorphism to $(\mathbb{Z}/\gcd(e_\ell,e)\mathbb{Z})^\times$. In particular, it is uniformly bounded. Proposition 3.3(ii) of *loc. cit.* then shows that the discrepancy of the ∞-adic orbit $\iota_\infty(K_\ell\cdot\omega_{\ell,1}^{r_{\ell,\zeta}})$ converges to 0 as $\ell\to+\infty$. Using standard results from measure theory like [Har98, Theorem 5.4] we deduce that

$$\lim_{\ell\to+\infty}\int_{S^1}\phi_\zeta\,d\delta_{\iota_\infty(K_\ell\cdot\omega_{\ell,1}^{r_{\ell,\zeta}})}=\int_{S^1}\phi_\zeta\,d\mathrm{Haar}=\int_{\mathbb{S}_{a,e}}\phi\,d\lambda_{a,e,\zeta}.$$

The statement then follows from (7.3). □

The next result is the analogue in our situation of Proposition 4.1, and gives the limit of the Archimedean local height corresponding to the torsion points in a strict sequence in $V_{a,e}$. To state it, consider the subset of $(\mathbb{R}/2\pi\mathbb{Z})^2$ defined as

$$\mathbb{D}_{a,e} = \bigcup_{j\in(\mathbb{Z}/e\mathbb{Z})^\times} \Big\{ u \in (\mathbb{R}/2\pi\mathbb{Z})^2 \,\Big|\, a_1u_1 + a_2u_2 = \frac{2\pi j}{e} \Big\}.$$

It is a union of $\varphi(e)$ parallel segments, and we denote by $\tau_{a,e}$ the probability measure on it induced from the Euclidean metric on these segments. Recall that

$$F\colon (\mathbb{C}^\times)^2 \to \mathbb{R}\cup\{-\infty\} \quad \text{and} \quad f\colon (\mathbb{R}/2\pi\mathbb{Z})^2 \to \mathbb{R}\cup\{-\infty\}$$

denote the functions defined in (4.1) and in (4.2), respectively.

Proposition 7.2. *Let $(\omega_\ell)_{\ell\geq 1}$ be a strict sequence in $V_{a,e}$ of nontrivial torsion points. Then*

$$\lim_{\ell\to+\infty} \mathrm{h}_\infty(\mathcal{P}(\omega_\ell)) = \int_{\mathbb{S}_{a,e}} F\, d\nu_{a,e} = \int_{\mathbb{D}_{a,e}} f\, d\tau_{a,e},$$

where $\mathcal{P}(\omega_\ell)$ is the vector of homogeneous coordinates of the projective point $C\cap\omega_\ell C$ as in (2.2).

Proof. For the first equality, note that when $e\neq 1$ the subset $\mathbb{S}_{a,e}$ does not contain the point $(1,1)$. In this case, the restriction of the function F to $\mathbb{S}_{a,e}$ is continuous, and therefore the statement follows from the formula in (4.6) and Proposition 7.1.

When $e=1$, the map $\psi\colon \mathbb{G}_{\mathrm{m}}(\overline{\mathbb{Q}}) \to V_{a,1}$ defined as $\psi(s) = (s^{-a_2}, s^{a_1})$ is an isomorphism of algebraic groups, because a is primitive. Denote also by $\psi\colon S^1 \to \mathbb{S}_{a,1}$ the restriction to the unit circle of the complex version of this map. Then the function

$$\psi^*F(s) - \log|s-1| = \log\max\Big(\frac{|s^{a_1}-s^{-a_2}|}{|s-1|}, \frac{|s^{a_1}-1|}{|s-1|}, \frac{|s^{-a_2}-1|}{|s-1|}\Big)$$

can be continuously extended to the point $s=1$. Thus the formula in (4.6) can be written as

$$\begin{aligned}\mathrm{h}_\infty(\mathcal{P}(\omega_\ell)) &= \int_{S^1} \psi^*F(s)\, d\delta_{O(\psi^{-1}(\omega_\ell))_\infty} \\ &= \int_{S^1} (\psi^*F(s) - \log|s-1|)\, d\delta_{O(\psi^{-1}(\omega_\ell))_\infty} + \int_{S^1} \log|s-1|\, d\delta_{O(\psi^{-1}(\omega_\ell))_\infty}.\end{aligned}$$

The first equality in the statement follows by applying the classical equidistribution theorem for roots of unity to the first integral in the formula above, and Lemma 2.4 to the second in a similar way as it was done to obtain the first equality in (4.9).

On the other hand, note that the image of $\mathbb{S}_{a,e}$ under the cotropicalization map coincides with the union of parallel segments $\mathbb{D}_{a,e}$ and that the direct image measure $\mathrm{cotrop}_*\,\nu_{a,e}$ coincides with $\tau_{a,e}$. The second equality is then a direct consequence of the first together with the change of variables formula. □

Set for short

$$\eta_{a,e} = \int_{\mathbb{D}_{a,e}} f\, d\tau_{a,e}. \tag{7.4}$$

We have the following result in the spirit of Theorem 6.1.

Theorem 7.3. *Let $(\omega_\ell)_{\ell\geq 1}$ be a strict sequence in $V_{a,e}$ of nontrivial torsion points. Then*

$$\lim_{\ell\to+\infty} \mathrm{h}(C\cap\omega_\ell C) = \eta_{a,e}.$$

Proof. This follows readily from the definition of the height together with Proposition 7.2 and Corollary 3.6. □

Using Theorem 7.3, we can find other interesting limit values for the height, as the next two examples show.

Example 7.4. For $a=(0,1)$ and $e=1$ the segment $\mathbb{D}_{a,e}$ can be parametrized with the unit interval by the map $w\mapsto(2\pi w,0)$. Hence

$$\eta_{a,e} = \int_0^1 \log\max(|\mathrm{e}^{\mathrm{i}2\pi w}-1|,|0|,|\mathrm{e}^{\mathrm{i}2\pi w}-1|)\,dw = \int_0^1 \log|\mathrm{e}^{\mathrm{i}2\pi w}-1|\,dw = 0.$$

By Theorem 7.3, the limit of the height of $C\cap\omega_\ell C$ for a strict sequence $(\omega_\ell)_{\ell\geq1}$ in $Z(t_2-1)$ of nontrivial torsion points is equal to 0, in agreement with Proposition 2.2. A similar observation holds for $a\in\{(1,0),(1,-1)\}$.

Example 7.5. For $a=(2,-1)$ and $e=1$ the segment $\mathbb{D}_{a,e}$ can be parametrized with the unit interval by the map $w\mapsto(2\pi w,4\pi w)$. Hence

$$\begin{aligned}\eta_{a,e} &= \int_0^1 \log\max(|\mathrm{e}^{\mathrm{i}4\pi w}-\mathrm{e}^{\mathrm{i}2\pi w}|,|\mathrm{e}^{\mathrm{i}4\pi w}-1|,|\mathrm{e}^{\mathrm{i}2\pi w}-1|)\,dw\\ &= \int_0^1 \log|\mathrm{e}^{\mathrm{i}2\pi w}-1|\,dw + \int_0^1 \log\max(1,|\mathrm{e}^{\mathrm{i}2\pi w}+1|)\,dw = \mathrm{m}(x_0+x_1+x_2),\end{aligned}$$

where the last equality follows from Jensen's formula. This logarithmic Mahler measure was computed by Smyth as

$$\mathrm{m}(x_0+x_1+x_2) = \frac{3\sqrt{3}}{4\pi}L(\chi_{-3},2) = 0.323065\ldots$$

for the L-function associated to odd Dirichlet character modulo 3 [Smy81]. By Theorem 7.3, this quantity gives the limit of the height of $C\cap\omega_\ell C$ for a strict sequence $(\omega_\ell)_{\ell\geq1}$ in $Z(t_1^2t_2^{-1}-1)$ of nontrivial torsion points, like that in in Example 2.1. A similar situation occurs when $a\in\{(1,1),(1,-2)\}$.

Similarly as in Section 6, we can extend our study to strict sequences of nonempty finite subsets of $V_{a,e}$, in the sense of Definition 6.4. Each algebraic subgroup of $\mathbb{G}_{\mathrm{m}}^2(\overline{\mathbb{Q}})$ not containing $V_{a,e}$ intersects this algebraic subset in finitely many points, and therefore any sequence of nonempty finite subsets of $V_{a,e}$ satisfying $\lim_{\ell\to+\infty}\#E_\ell=+\infty$ is automatically strict in $V_{a,e}$ (the condition is not necessary, though).

Example 7.6. The sequence of nonempty subsets of d-torsion points of $V_{a,e}$ is strict. Indeed, let $d\geq1$ and consider the monomial map $\chi^a\colon\mu_d^2\to\mu_d$, so that

$$\mu_d^2\cap V_{a,e} = (\chi^a)^{-1}(\mu_d\cap\mu_e^\circ). \tag{7.5}$$

If $e\nmid d$ then $\mu_d\cap\mu_e^\circ=\emptyset$ and so $\mu_d^2\cap V_{a,e}=\emptyset$. Otherwise $e\mid d$ and $\mu_e^\circ\subset\mu_d$, and so it follows from (7.5) and the surjectivity of χ^a that

$$\#(\mu_d^2\cap V_{a,e}) = \#\operatorname{Ker}(\chi^a)\cdot\#\mu_e^\circ = d\,\varphi(e).$$

Thus $(\mu_d^2\cap V_{a,e})_{d\geq1,e\mid d}$ is a strict sequence in $V_{a,e}$.

The next two results are the analogues in our current setting of Theorem 6.6 and Corollary 6.8. The first is a direct consequence of Lemma 6.7 and Theorem 7.3, and the second follows using Example 7.6.

Theorem 7.7. *Let $(W_\ell)_{\ell\geq 1}$ be a strict sequence of nonempty finite subsets of $V_{a,e}$ of nontrivial torsion points. Then for each $\varepsilon > 0$ we have that*

$$\lim_{\ell\to+\infty} \frac{\#\{\omega \in W_\ell \mid |\,\mathrm{h}(C\cap\omega C) - \eta_{a,e}| < \varepsilon\}}{\#W_\ell} = 1.$$

Moreover $\displaystyle\lim_{\ell\to+\infty} \frac{1}{\#W_\ell} \sum_{\omega\in W_\ell} \mathrm{h}(C\cap\omega C) = \eta_{a,e}.$

Corollary 7.8. *For each $\varepsilon > 0$ we have that*

$$\lim_{\substack{d\to+\infty\\ e\mid d}} \frac{\#\{\omega \in \mu_d^2\cap V_{a,e}\setminus\{(1,1)\} \mid |\,\mathrm{h}(C\cap\omega C) - \eta_{a,e}| < \varepsilon\}}{\#(\mu_d^2\cap V_{a,e}\setminus\{(1,1)\})} = 1.$$

Moreover $\displaystyle\lim_{\substack{d\to+\infty\\ e\mid d}} \frac{1}{\#(\mu_d^2\cap V_{a,e}\setminus\{(1,1)\})} \sum_{\omega\in\mu_d^2\cap V_{a,e}\setminus\{(1,1)\}} \mathrm{h}(C\cap\omega C) = \eta_{a,e}.$

8. Visualizing the results

In this section we present a series of computations done with the `SageMath` notebook [GS22] that allow to visualize our results while at the same time suggest further intriguing questions and conjectures.

We focus on the height values of the points of the form $P(\omega) = C\cap\omega C$ as ω ranges in the set of nontrivial d-torsion points of $\mathbb{G}_{\mathrm{m}}^2(\overline{\mathbb{Q}})$. The next statement specifies how to enumerate these torsion points and compute the corresponding heights.

Proposition 8.1. *Let $d \geq 1$ and $\zeta \in \mu_d^\circ$. Then*

(1) the map $(\mathbb{Z}/d\mathbb{Z})^2 \to \mu_d^2$ defined as $c \mapsto (\zeta^{c_1}, \zeta^{c_2})$ is a bijection,

(2) for each $c \in (\mathbb{Z}/d\mathbb{Z})^2 \setminus \{(0,0)\}$, letting $e = d/\gcd(c_1, c_2, d)$ we have that

$$\begin{aligned}\mathrm{h}(C\cap(\zeta^{c_1},\zeta^{c_2})C) = &-\frac{\Lambda(e)}{\varphi(e)}\\ &+\frac{1}{\varphi(e)}\sum_{k\in(\mathbb{Z}/e\mathbb{Z})^\times} \log\max(|\mathrm{e}^{2\pi\mathrm{i}c_2k/d} - \mathrm{e}^{2\pi\mathrm{i}c_1k/d}|, |\mathrm{e}^{2\pi\mathrm{i}c_2k/d} - 1|, |\mathrm{e}^{2\pi\mathrm{i}c_1k/d} - 1|)\end{aligned}$$

where Λ and φ denote the von Mangoldt and the Euler totient functions,

(3) the function $(\mathbb{Z}/d\mathbb{Z})^2 \setminus \{(0,0)\} \to \mathbb{R}$ defined as $(c_1, c_2) \mapsto \mathrm{h}(C\cap(\zeta^{c_1},\zeta^{c_2})C)$ does not depend on the choice of $\zeta \in \mu_d^\circ$ and it is invariant under the transformations

$$(c_1, c_2) \mapsto (c_2, c_1), \quad (c_1, c_2) \mapsto (-c_2, c_1 - c_2), \quad (c_1, c_2) \mapsto (-c_1, -c_2)$$

on $(\mathbb{Z}/d\mathbb{Z})^2 \setminus \{(0,0)\}$.

Proof. The statement in (1) is a particular case of (1.1), whereas that in (2) follows directly from Lemma 2.3, Corollary 3.4 and the definition of the height. Finally, the statement in (3) is an easy consequence of that in (2). □

Displaying the computations. For each $d \geq 1$ we can numerically compute the height of the point $C \cap \omega C$ for each $\omega \in \mu_d^2 \setminus \{(1,1)\}$ using the formula in Proposition 8.1(2), and display the outputs in two images. The first shows the obtained height values as an unordered plot on the unit interval, whereas the second represents them in a meaningful and organized way in the unit square.

The latter image is produced as we next explain. Choosing $\zeta \in \mu_d^\circ$, Proposition 8.1(1) identifies μ_d^2 with the set of grid points $\{0, 1/d, \ldots, (d-1)/d\}^2$ of the unit square $[0,1)^2$. To visualize the behavior of the function

$$\omega \mapsto \mathrm{h}(C \cap \omega C) \quad \text{for } \omega \in \mu_d^2 \setminus \{(1,1)\}$$

we subdivide this square into d^2-many cells centered at these grid points. Apart from that with center $(0,0)$, each of these cells corresponds to a nontrivial d-torsion point ω, and we color it with a tone of gray that is as dark as the height of $C \cap \omega C$ is larger within the range $[0, \log(2)]$. By virtue of Proposition 8.1(3), the resulting image does not depend on the choice of the primitive d-root of unity ζ.

Figure 8.1 shows the resulting images for $d = 120$. For future considerations, we enrich the first plot with the special values

$$\eta = \frac{2\,\zeta(3)}{3\,\zeta(2)} = 0.487175\ldots \quad \text{and} \quad \theta = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) = 0.323065\ldots \tag{8.1}$$

marked with a red line and an orange line, respectively.

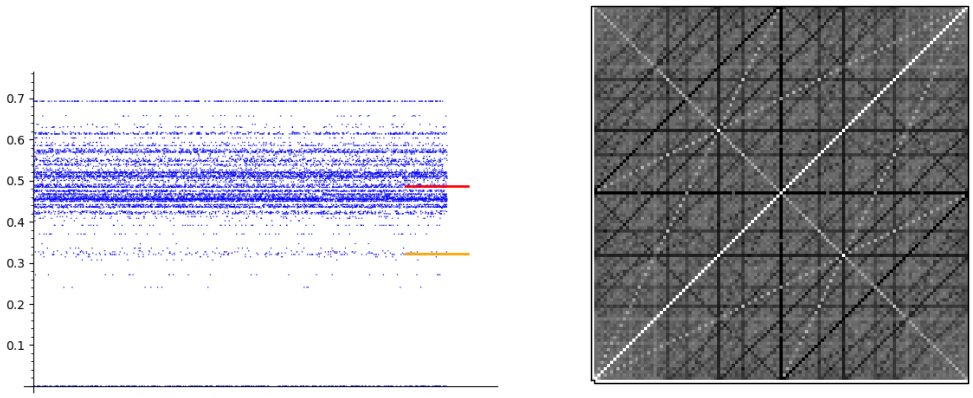

FIGURE 8.1. Distribution of heights associated to d-torsion points, $d = 120$

Bounds and symmetries. These figures allow to appreciate several features of the height values we study.

As predicted by Proposition 2.2, the extremal values are 0 and $\log(2)$, as can be seen in the left image of Figure 8.1. In the right image, the minimal height (in white) is reached for the torsion points corresponding to the cells in the horizontal, diagonal and vertical lines passing through the origin of the square, plus at the grid points $(1/3, 2/3)$ and $(2/3, 1/3)$ whenever $3 \mid d$. The maximal height (in black) is only attained when d is even but not a power of 2. When this is the case, it is visible on the horizontal and vertical lines through the center of the square, and on the diagonal passing through the midpoint of an edge of the square, excluding the cells centered at the grid points of the form $(2^{-k} b_1, 2^{-k} b_2)$ with $b_1, b_2 \in \mathbb{Z}$ and $k \geq 0$.

This right image also makes apparent the invariance of the height values under the transformations in Proposition 8.1(3) and their compositions. Up to rescaling, these are the same automorphisms on $(\mathbb{R}/2\pi\mathbb{Z})^2$ in Proposition 5.2, which explains the analogy of this image with the right image in Figure 5.1. In particular, the triangle with vertices $(0,0)$, $(1/2,0)$ and $(2/3,1/3)$ is a fundamental domain for the height values.

Limit value. The limit behavior of the height is also visible. As predicted by Corollary 6.8, most of the points in the left image of Figure 8.1 cluster around the value indicated by the red line, and most of the cells in the right image have a tone of gray that approaches the intensity

$$\frac{\eta}{\log(2)} = 0.702845\ldots$$

as $d \to +\infty$.

We can make this phenomenon more visible with a couple of plots by taking $\varepsilon > 0$ and computing the quantities

$$\frac{1}{d^2-1}\#\{\omega \in \mu_d^2 \setminus \{(1,1)\} \mid |\mathrm{h}(C \cap \omega C) - \eta| < \varepsilon\} \quad \text{and} \quad \frac{1}{d^2-1}\sum_{\omega \in \mu_d^2 \setminus \{(1,1)\}} \mathrm{h}(C \cap \omega C)$$

for a family of values of d. Figure 8.2 collects them for all $1 \leq d \leq 250$ and $\varepsilon = 0.1$.

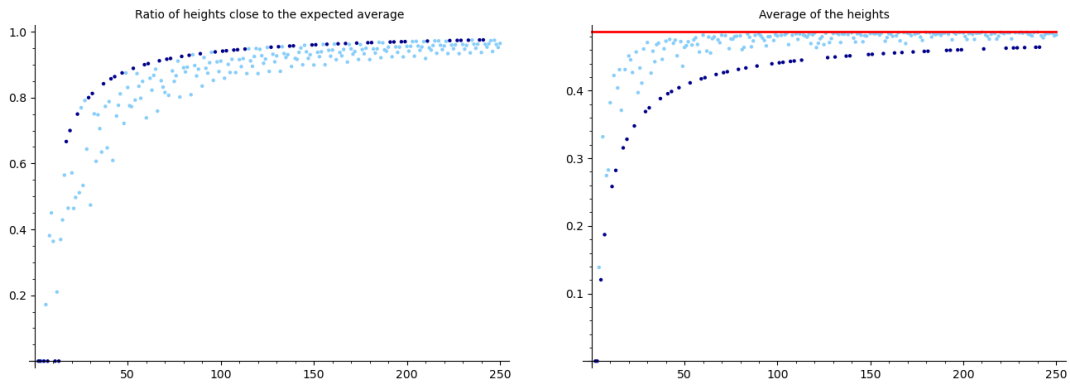

FIGURE 8.2. Ratio of heights for d-torsion points around the limit and their mean, $1 \leq d \leq 250$, with marked prime values

In accordance with Corollary 6.8, these plots confirm that most of the considered heights cluster near η as $d \to +\infty$, and that their mean converges to this limit value, marked in red in the image on the right.

It would be interesting to understand other patterns suggested by these computations. For instance, the right plot of Figure 8.2 hints to a positive answer to the following question.

Question 8.2. *Does it hold that* $\displaystyle\frac{1}{d^2-1}\sum_{\omega \in \mu_d^2 \setminus \{(1,1)\}} \mathrm{h}(C \cap \omega C) < \eta$ *for all* $d \geq 1$*?*

Torsion curves. We can also visualize the asymptotics obtained in Section 7. Recall that for a primitive vector $a \in \mathbb{Z}^2$ and a positive integer e we consider the disjoint

union of torsion curves

$$V_{a,e} = \bigcup_{\zeta \in \mu_e^\circ} Z(\chi^a - \zeta) \subset \mathbb{G}_{\mathrm{m}}^2(\overline{\mathbb{Q}}),$$

and that the value $\eta_{a,e}$ in (7.4) is the limit of the heights corresponding to strict sequences of torsion points in this algebraic subset.

Corollary 7.8 becomes apparent in the right image of Figure 8.1, which shows that most of the height values for the cells in the union of segments

$$\bigcup_{j \in (\mathbb{Z}/e\mathbb{Z})^\times} \left\{ \left(\frac{c_1}{d}, \frac{c_2}{d} \right) \,\middle|\, \frac{a_1 c_1 + a_2 c_2}{d} = \frac{j}{e} \right\}$$

cluster around $\eta_{a,e}$ as $d \to +\infty$ with $e \mid d$. Indeed, the left image in this figure plus a bit of imagination allows to see how these limit values are approached, and it would not be difficult to produce plots analogous to those in Figure 8.2 for a given algebraic subset $V_{a,e}$.

Small heights. By Zhang's theorem proving the toric Bogomolov conjecture, the height of the nontorsion points of the projective line C is bounded below by a positive constant, see [Zag93] for an effective version. This can be appreciated in the left image of Figure 8.1 as a gap between 0 and the rest of the values.

A closer examination suggests that the first accumulation point for the heights under consideration is the constant θ in (8.1). As shown in Example 7.5, this value is approached by the heights corresponding to the grid points in the three segments through the origin and orthogonal to the vectors $(2, -1)$, $(1, 1)$ and $(1, -2)$. We can turn this observation into a formal question as follows.

Question 8.3. *Let $\varepsilon > 0$. Is the set $\{\omega \in \mu_\infty^2 \setminus \{(1,1)\} \mid 0 < \mathrm{h}(C \cap \omega C) \leq \theta - \varepsilon\}$ finite?*

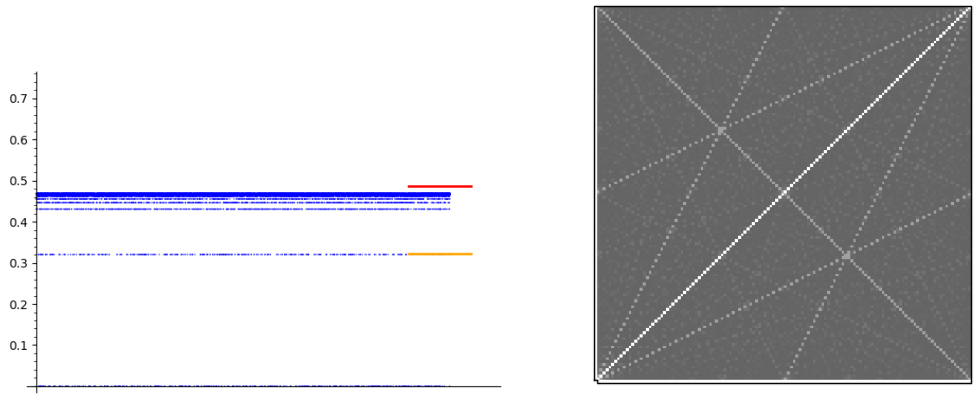

FIGURE 8.3. Distribution of heights associated to d-torsion points, $d = 131$

Prime orders. In Figure 8.2, the quantities corresponding to prime values of d are marked in dark blue, and they seem to have a regular behavior for the two considered features. Indeed, both the ratio of height values near η and their mean appear to follow hyperbola-like patterns, at least for d large enough. Moreover, these patterns seem to bound, respectively from above and from below, the corresponding quantity for a general d. It would be interesting to explain these behaviors.

In fact, when d is prime, even the single height values corresponding to d-torsion points seem to be simpler and tamer than those for composite d, as can be seen in Figure 8.3 for $d = 131$.

This figure and similar ones suggest that for every prime d all the height values corresponding to nontrivial d-torsion points lie below η, and that when they are nonzero, they are at least as large as θ. In precise terms:

Question 8.4. *Let p be a prime and $\omega \in \mu_p^2 \setminus \{(1,1)\}$ with $\mathrm{h}(C \cap \omega C) \neq 0$. Does it hold that $\theta \leq \mathrm{h}(C \cap \omega C) < \eta$?*

PART II

In this part we present an interpretation of Theorem 6.1 from the viewpoint of Arakelov geometry, that allows to recover it in a more intrinsic way using the interplay between arithmetic and convex objects from the Arakelov geometry of toric varieties.

The first main result here is Theorem 10.4, linking the limit height in Theorem 6.1 with the Arakelov height of the subscheme of $\mathbb{P}^2_{\mathbb{Z}}$ defined by the linear polynomial $x_0 + x_1 + x_2$ with respect to suitable metrized line bundles. The second main result is Corollary 13.3, showing that this height agrees with the average of a piecewise linear function over the Archimedean amoeba of this subscheme, that can in turn be computed as a rational multiple of a quotient of special values of the Riemann zeta function.

The treatment of this second part of the article is less elementary and certainly not self-contained, building on the theory developed by Gillet and Soulé [GS90] and its refinement by Maillot [Mai00], and on its study in the toric situation by Burgos Gil, Philippon and the second named author [BPS14] and by the first named author [Gua18b]. We will recall the main objects and results that we will use, assuming that the reader has some working knowledge of these subjects, including the basics of complex analytic geometry and of integral models of schemes.

9. Semipositive metrics in complex geometry

Let X be a projective complex manifold with sheaf of holomorphic functions O_X. Let L be a holomorphic line bundle on X, that is, a locally free sheaf of O_X-modules of rank 1. A *(continuous) metric* $\|\cdot\|$ on L is a rule that to every open subset U of X and every section s of L on U assigns a continuous function

$$\|s(\cdot)\| \colon U \to \mathbb{R}_{\geq 0} \tag{9.1}$$

that is compatible with restrictions to smaller open sets, and verifies that

(1) for every $p \in U$ we have that $\|s(p)\| = 0$ if and only if $s(p) = 0$,
(2) for every $p \in U$ and $\lambda \in O_X(U)$ we have that $\|(\lambda s)(p)\| = |\lambda(p)|\,\|s(p)\|$.

The pair $\overline{L} = (L, \|\cdot\|)$ is called a *metrized line bundle* on X.

Remark 9.1. To define a metric on L, it is enough to give a compatible choice of functions as those in (9.1) for a family of nonvanishing sections of L on open subsets that cover the whole of X.

To a metrized line bundle $\overline{L}$ on X one can associate its *first Chern current* $\mathrm{c}_1(\overline{L})$, which is the current of bidegree $(1,1)$ defined on any open subset U of X as

$$\mathrm{c}_1(\overline{L})\big|_U = dd^{\mathrm{c}}\big[-\log\|s\|\big|_U\big]$$

for any nonvanishing section s on U, where for a real-valued function f we denote by $[f]$ the associated distribution, and the operators d and d^{c} are taken in the sense of currents. This current does not depend on the choice of the section because of the Lelong–Poincaré formula.

A metrized line bundle $\overline{L}$ is *semipositive* (respectively *smooth*) if for every nonvanishing local section s the function $p \mapsto -\log \|s(p)\|$ is plurisubharmonic (respectively smooth).

Example 9.2. The trivial line bundle O_X admits a *trivial metric* $\|\cdot\|_{\mathrm{tr}}$, which is that defined by setting, for each holomorphic function λ on an open subset U of X,

$$\|\lambda(p)\|_{\mathrm{tr}} = |\lambda(p)| \quad \text{for all } p \in U.$$

The corresponding metrized line bundle is denoted by $\overline{O_X}^{\mathrm{tr}}$. Computing its first Chern current using the holomorphic function $\lambda = 1$ we verify that $\mathrm{c}_1(\overline{O_X}^{\mathrm{tr}}) = 0$, and so this metrized line bundle is semipositive.

Example 9.3. Let $O(1)$ be the hyperplane line bundle of the complex projective space $\mathbb{P}^n(\mathbb{C})$. There is a one-to-one correspondence between its global sections and the linear polynomials in the homogeneous coordinates of $\mathbb{P}^n(\mathbb{C})$. For each global section s we denote by l_s the corresponding linear polynomial and then set

$$\|s(p)\|_{\mathrm{can}} = \frac{|l_s(p_0, \dots, p_n)|}{\max(|p_0|, \dots, |p_n|)} \quad \text{for all } p = [p_0 : \cdots : p_n] \in \mathbb{P}^n(\mathbb{C}).$$

Since $O(1)$ is generated by its global sections, this assignment determines through Remark 9.1 a metric on this line bundle, which is called the *canonical metric* of $O(1)$. The corresponding metrized line bundle is denoted by $\overline{O(1)}^{\mathrm{can}}$. It is semipositive [BPS14, Example 1.4.4] but not smooth.

Example 9.4. Consider the function $R\colon \mathbb{R}^{n+1} \setminus \{(0, \dots, 0)\} \to \mathbb{R}_{>0}$ defined as

$$R(w) = \exp\Big(\int_{(S^1)^n} \log |w_0 + z_1 w_1 + \cdots + z_n w_n| \, d\nu_n(z) \Big),$$

where $(S^1)^n$ denotes the compact torus of $(\mathbb{C}^\times)^n$ and ν_n its probability Haar measure. Its value at a given point is the Mahler measure of an affine polynomial, and so it is a well-defined real number.

The function R is continuous and positive homogeneous of degree 1. Similarly as for the canonical metric in the preceding example, the *Ronkin metric* of $O(1)$ is obtained by setting, for each global section s,

$$\|s(p)\|_{\mathrm{Ron}} = \frac{|l_s(p_0, \dots, p_n)|}{R(|p_0|, \dots, |p_n|)} \quad \text{for all } p = [p_0 : \cdots : p_n] \in \mathbb{P}^n(\mathbb{C}).$$

The corresponding metrized line bundle is denoted by $\overline{O(1)}^{\mathrm{Ron}}$.

Remark 9.5. Ronkin metrics on line bundles on toric varieties were introduced in [Gua18b], and they can be defined from any choice of nonzero Laurent polynomial compatible with the fan of the toric variety. The metric in Example 9.4 agrees with that from Definition 5.5 of *loc. cit.* for the Laurent polynomial $1 + t_1 + \cdots + t_n$. As explained therein, it is semipositive.

Given a pure d-dimensional analytic cycle Z of X, we denote by δ_Z its associated integration current and by $|Z|$ its support. Let also $\overline{L_1}, \dots, \overline{L_d}$ be a family of semipositive metrized line bundles on X. Then, Bedford–Taylor's theory allows to construct a measure on X, denoted by

$$\mathrm{c}_1(\overline{L_1}) \wedge \ldots \wedge \mathrm{c}_1(\overline{L_d}) \wedge \delta_Z$$

and called the *complex Monge–Ampère measure* of $\overline{L_1}, \dots, \overline{L_d}$ and Z, see [Dem12, Sections III.3 and III.4] for its definition and basic properties. This measure is supported on $|Z|$, and has finite integral against functions with at most logarithmic singularities along analytic subsets of $|Z|$ of lower dimension. In particular, it gives zero mass to those analytic subsets. Finally, this measure is positive whenever Z is effective.

The next proposition is a particular case of well-known results in analytic geometry, see for instance [BE21, Lemma 8.17(ii)]. It is also a direct consequence of the commutativity of the $*$-product from Arakelov geometry shown in [GS90, Corollary 2.2.9] and extended in [Mai00, Proposition 5.3.6] to metrized line bundles that are not necessarily smooth. We include its proof for convenience and greater clarity.

Proposition 9.6 (metric Weil reciprocity law)**.** *Let X be a smooth projective complex curve. For $i = 1, 2$, let also $\overline{L_i} = (L_i, \|\cdot\|_i)$ be a semipositive metrized line bundle on X, and s_i a nonzero rational section of L_i. Suppose that the 0-cycles*

$$Z_i = \sum_{p \in X} \mathrm{ord}_p(s_i)\,[p] \qquad \textit{for } i = 1, 2$$

have disjoint supports. Then

$$\int_X \log \|s_1\|_1 \left(\mathrm{c}_1(\overline{L_2}) \wedge \delta_X - \delta_{Z_2}\right) = \int_X \log \|s_2\|_2 \left(\mathrm{c}_1(\overline{L_1}) \wedge \delta_X - \delta_{Z_1}\right). \tag{9.2}$$

Proof. Both sides of the formula in (9.2) are linear in the choice of the metrized line bundles and rational sections. Hence we can assume that L_1 and L_2 are ample. When this is the case, their metrics can be uniformly approached by smooth semipositive metrics on the same line bundles. Since both sides of this formula are continuous with respect to the uniform convergence of metrics, we can then reduce to the case in which both $\|\cdot\|_1$ and $\|\cdot\|_2$ are smooth. We suppose this for the rest of the proof.

For each $p \in |Z_1| \cup |Z_2|$ choose a closed neighborhood $B_p \subset X$ with smooth boundary. Assume that all of these neighborhoods are disjoint and consider the open subset with smooth boundary

$$U = X \setminus \bigcup_{p \in |Z_1| \cup |Z_2|} B_p.$$

Set $f_i = -\log \|s_i\|_i$ for each i. Since the rational section s_i is regular and non-vanishing on U and the metric $\|\cdot\|_i$ is smooth, the restriction to U of the complex Monge–Ampère measure $\mathrm{c}_1(\overline{L_i}) \wedge \delta_X$ is given by the $(1,1)$-form $dd^{\mathrm{c}} f_i|_U$. By [Dem12, Chapter III, Formula 3.1],

$$\begin{aligned} \int_U \log \|s_1\|_1\, \mathrm{c}_1(\overline{L_2}) \wedge \delta_X - \log \|s_2\|_2\, \mathrm{c}_1(\overline{L_1}) \wedge \delta_X \\ = \int_U f_2\, dd^{\mathrm{c}} f_1 - f_1\, dd^{\mathrm{c}} f_2 = \int_{\partial U} f_2\, d^{\mathrm{c}} f_1 - f_1\, d^{\mathrm{c}} f_2. \end{aligned} \tag{9.3}$$

Note that the integral in the left-hand side of (9.3) approaches the same integral over the whole of X when all the considered neighborhoods get arbitrarily small.

By spelling out the different connected components of ∂U we obtain that

$$\int_{\partial U} f_1\, d^{\mathrm{c}} f_2 = -\sum_{p\in|Z_1|}\int_{\partial B_p} f_1\, d^{\mathrm{c}} f_2 - \sum_{p\in|Z_2|}\int_{\partial B_p} f_1\, d^{\mathrm{c}} f_2. \tag{9.4}$$

The first sum in the right-hand side of (9.4) tends to 0 when these neighborhoods get small, as $d^{\mathrm{c}} f_2$ is a smooth $(0,1)$-form on B_p for every $p \in |Z_1|$ and f_1 has a logarithmic singularity at every such point.

Now let $p \in |Z_2|$. Choosing the neighborhood B_p appropriately, we can suppose that there is a closed disc $V \subset \mathbb{C}$ centered at the origin together with a biholomorphic map $\varphi\colon V \to B_p$ such that $\varphi(0) = p$. Then $\|(s_2 \circ \varphi)(z)\|_2 = |z|^m h(z)$ for $m = \mathrm{ord}_p(s_2)$ and a nonvanishing smooth function h. Hence

$$\begin{aligned} -\int_{\partial B_p} f_1\, d^{\mathrm{c}} f_2 &= \int_{\partial V} f_1 \circ \varphi\; d^{\mathrm{c}} \log \|s_2 \circ \varphi\|_2 \\ &= \frac{m}{2}\int_{\partial V} f_1 \circ \varphi\; d^{\mathrm{c}} \log(z\overline{z}) + \int_{\partial V} f_1 \circ \varphi\; d^{\mathrm{c}} \log(h). \end{aligned} \tag{9.5}$$

Since $d^{\mathrm{c}} \log(h)$ is a smooth $(0,1)$-form on V, the second summand in the right-hand side of (9.5) tends to 0 when B_p shrinks to the point p. For the first summand, recall that $d^{\mathrm{c}} = (\partial - \overline{\partial})/2\pi\mathrm{i}$ with ∂ and $\overline{\partial}$ the Dolbeault operators. Hence

$$d^{\mathrm{c}} \log(z\overline{z}) = \frac{1}{2\pi\mathrm{i}}(\partial \log(z) - \overline{\partial} \log(\overline{z})) = \frac{1}{2\pi\mathrm{i}}\Big(\frac{dz}{z} - \frac{d\overline{z}}{\overline{z}}\Big),$$

and noting that the function f_1 is real-valued we deduce that

$$\int_{\partial V} f_1 \circ \varphi\; d^{\mathrm{c}} \log(z\overline{z}) = \frac{1}{2\pi\mathrm{i}}\int_{\partial V} f_1 \circ \varphi\, \Big(\frac{dz}{z} - \frac{d\overline{z}}{\overline{z}}\Big) = \mathrm{Re}\,\Big(\frac{1}{\pi\mathrm{i}}\int_{\partial V} f_1 \circ \varphi\, \frac{dz}{z}\Big). \tag{9.6}$$

Since B_p contains no point of $|Z_1|$, the function $f_1 \circ \varphi$ is smooth on V. Then Cauchy's formula [Dem12, Chapter I, Formula 3.2] gives

$$\frac{1}{2\pi\mathrm{i}}\int_{\partial V} f_1 \circ \varphi\, \frac{dz}{z} = (f_1 \circ \varphi)(0) + \int_V \frac{1}{\pi z}\frac{\partial f_1 \circ \varphi}{\partial \overline{z}}\, d\lambda(z), \tag{9.7}$$

where λ denotes the Lebesgue measure of $\mathbb{C}$. As B_p shrinks to p, the integral in the right-hand side of (9.7) converges to 0 because the function $z \mapsto 1/z$ is integrable on V, and so the integral in the left-hand side tends to $(f_1 \circ \varphi)(0) = f_1(p)$. From (9.5) and (9.6) we deduce that $-\int_{\partial B_p} f_1\, d^{\mathrm{c}} f_2$ converges to $\mathrm{ord}_p(s_2) f_1(p)$.

Thus when all the considered neighborhoods get small, it follows from (9.4) that the integral $\int_{\partial U} f_1\, d^{\mathrm{c}} f_2$ converges to the quantity

$$\sum_{p\in|Z_2|} \mathrm{ord}_p(s_2) f_1(p) = -\int_X \log \|s_1\|\, \delta_{Z_2}.$$

A similar consideration shows that $\int_{\partial U} f_2\, d^{\mathrm{c}} f_1$ converges to $-\int_X \log \|s_2\|\, \delta_{Z_1}$. Taking the limit when the union of these neighborhoods converges to $|Z_1| \cup |Z_2|$, we deduce from (9.3) that

$$\int_X \log \|s_1\|_1\, \mathrm{c}_1\big(\overline{L_2}\big) \wedge \delta_X - \log \|s_2\|_2\, \mathrm{c}_1\big(\overline{L_1}\big) \wedge \delta_X = \int_X \log \|s_1\|\, \delta_{Z_2} - \int_X \log \|s_2\|\, \delta_{Z_1},$$

as stated. □

Remark 9.7. The classical Weil reciprocity law is the equality

$$\prod_{p\in X} f_1(p)^{\mathrm{ord}_p(f_2)} = \prod_{p\in X} f_2(p)^{\mathrm{ord}_p(f_1)} \tag{9.8}$$

for any pair of nonzero rational functions f_1 and f_2 of a smooth projective curve X whose associated 0-cycles have disjoint support [Sil09, Exercise 2.11].

Proposition 9.6 can be seen as a metric version of this reciprocity law, that essentially contains it. Indeed choosing both $\overline{L_1}$ and $\overline{L_2}$ as the trivial metrized line bundle of X from Example 9.2, this proposition yields the equality between the absolute values of the right-hand and left-hand sides of (9.8).

10. The limit height of the intersection as an Arakelov height

In this section we link Theorem 6.1 to the theory of heights from Arakelov geometry. To this end, we first recall the basic elements of this theory for arithmetic varieties equipped with semipositive metrized line bundles that are not necessarily smooth.

Let $\mathscr{X}$ be an *arithmetic variety*, that is a regular integral projective flat scheme over $\mathrm{Spec}(\mathbb{Z})$, with sheaf of regular functions $\mathscr{O}_{\mathscr{X}}$. By the regularity assumption, its set of complex points $\mathscr{X}(\mathbb{C})$ is a projective complex manifold. Given a line bundle $\mathscr{L}$ on $\mathscr{X}$, its analytification $\mathscr{L}^{\mathrm{an}}$ is a holomorphic line bundle on $\mathscr{X}(\mathbb{C})$.

A *semipositive metrized line bundle* on $\mathscr{X}$ is a pair

$$\overline{\mathscr{L}} = (\mathscr{L}, \|\cdot\|)$$

where $\mathscr{L}$ is a line bundle on $\mathscr{X}$ and $\|\cdot\|$ a semipositive metric on $\mathscr{L}^{\mathrm{an}}$ that is invariant under the involution on $\mathscr{X}(\mathbb{C})$ induced by the complex conjugation. We denote by $\overline{\mathscr{L}^{\mathrm{an}}} = (\mathscr{L}^{\mathrm{an}}, \|\cdot\|)$ the associated semipositive metrized line bundle on $\mathscr{X}(\mathbb{C})$.

For a pure d-dimensional cycle $\mathscr{Z}$ of $\mathscr{X}$ and a family of semipositive metrized line bundles $\overline{\mathscr{L}_i} = (\mathscr{L}_i, \|\cdot\|_i)$ for $i = 1, \dots, d$, on $\mathscr{X}$, we define its *(Arakelov) height*

$$\mathrm{h}_{\overline{\mathscr{L}_1},\dots,\overline{\mathscr{L}_d}}(\mathscr{Z})$$

by means of the following recursion on the dimension:

(1) when $d = 0$, if $\mathscr{Z}$ is a prime cycle then it is an integral closed point of $\mathscr{X}$ and its function field $\mathrm{K}(\mathscr{Z})$ is finite. In this case its height is defined as

$$\mathrm{h}(\mathscr{Z}) = \log(\#\,\mathrm{K}(\mathscr{Z})).$$

In the general case, the height of $\mathscr{Z}$ is defined by linearity.

(2) when $d \geq 1$, pick a rational section s of $\mathscr{L}_d$ that is regular and nonvanishing on a dense open subset of the support of $\mathscr{Z}$ and set

$$\mathrm{h}_{\overline{\mathscr{L}_1},\dots,\overline{\mathscr{L}_d}}(\mathscr{Z}) = \mathrm{h}_{\overline{\mathscr{L}_1},\dots,\overline{\mathscr{L}_{d-1}}}(\mathrm{div}(s)\cdot\mathscr{Z}) - \int_{\mathscr{X}(\mathbb{C})} \log\|s^{\mathrm{an}}\|_d \; \mathrm{c}_1\big(\overline{\mathscr{L}_1^{\mathrm{an}}}\big)\wedge\ldots\wedge\mathrm{c}_1\big(\overline{\mathscr{L}_{d-1}^{\mathrm{an}}}\big)\wedge\delta_{\mathscr{Z}(\mathbb{C})},$$

where $\mathrm{div}(s)\cdot\mathscr{Z}$ denotes the intersection product of the Cartier divisor $\mathrm{div}(s)$ and the cycle $\mathscr{Z}$, and s^{an} stands for s considered as a meromorphic section of $\mathscr{L}_d^{\mathrm{an}}$.

The height is a real number that does not depend on the choice of the rational section s in (2) nor on the order in which the metrized line bundles are chosen.

Remark 10.1. The Arakelov height was introduced by Bost, Gillet and Soulé for smooth metrics through arithmetic intersection theory [BGS94], and later extended by Maillot to semipositive metrics that are not necessarily smooth [Mai00].

Example 10.2. Let $\mathbb{P}^n_{\mathbb{Z}}$ be the projective space over $\operatorname{Spec}(\mathbb{Z})$. The pair

$$\overline{\mathscr{O}(1)}^{\mathrm{can}} = (\mathscr{O}(1), \|\cdot\|_{\mathrm{can}})$$

where $\mathscr{O}(1)$ denotes the hyperplane line bundle of $\mathbb{P}^n_{\mathbb{Z}}$ and $\|\cdot\|_{\mathrm{can}}$ the canonical metric of the holomorphic line bundle $O(1)$ from Example 9.3 is a semipositive metrized line bundle on this arithmetic variety, called the *canonical metrized line bundle* of $\mathbb{P}^n_{\mathbb{Z}}$.

Let $\mathbb{P}^n(\overline{\mathbb{Q}})$ be the projective space introduced in Section 1. A point $\xi \in \mathbb{P}^n(\overline{\mathbb{Q}})$ with rational homogeneous coordinates can be identified with a $\mathbb{Q}$-point of $\mathbb{P}^n_{\mathbb{Z}}$ or equivalently with a scheme-theoretic point in its generic fiber. Denote by $\overline{\xi}$ its closure in $\mathbb{P}^n_{\mathbb{Z}}$, which is an integral subscheme of dimension 1. Then

$$\mathrm{h}_{\overline{\mathscr{O}(1)}^{\mathrm{can}}}(\overline{\xi}) = \mathrm{h}(\xi), \tag{10.1}$$

the quantity on the right-hand side being the height of ξ in the sense of (1.6). Proving this equality is a nice exercise that can be solved by unwrapping the corresponding definitions and by using Remark 1.1.

Remark 10.3. More generally, the Arakelov height can be defined for cycles of $\mathscr{X}_{\overline{\mathbb{Z}}}$, the base change of $\mathscr{X}$ with respect to the integral closure of the ring of integers. In this setting, a point $\xi \in \mathbb{P}^n(\overline{\mathbb{Q}})$ can be identified with a $\overline{\mathbb{Q}}$-point of $\mathbb{P}^n_{\overline{\mathbb{Z}}}$ or equivalently with a scheme-theoretic point in the generic fiber of $\mathbb{P}^n_{\overline{\mathbb{Z}}}$. The equality in (10.1) extends then to ξ and its closure $\overline{\xi}$ in $\mathbb{P}^n_{\overline{\mathbb{Z}}}$.

Now consider the semipositive metrized line bundles on $\mathbb{P}^2_{\mathbb{Z}}$

$$\overline{\mathscr{O}(1)}^{\mathrm{can}} = (\mathscr{O}(1), \|\cdot\|_{\mathrm{can}}) \quad \text{and} \quad \overline{\mathscr{O}(1)}^{\mathrm{Ron}} = (\mathscr{O}(1), \|\cdot\|_{\mathrm{Ron}}) \tag{10.2}$$

obtained by equipping the holomorphic line bundle $O(1)$ on $\mathbb{P}^2(\mathbb{C})$ with the canonical and the Ronkin metrics from Examples 9.3 and 9.4, respectively. Consider also the subscheme $\mathscr{C}$ of $\mathbb{P}^2_{\mathbb{Z}}$ defined by the homogeneous linear polynomial $x_0 + x_1 + x_2$. The line $C \subset \mathbb{P}^2(\overline{\mathbb{Q}})$ studied through Part I coincides with the set of $\overline{\mathbb{Q}}$-points of $\mathscr{C}$.

The following is our main result in this section.

Theorem 10.4. *Let $(\omega_\ell)_{\ell\geq 1}$ be a strict sequence in $\mathbb{G}^2_{\mathrm{m}}(\overline{\mathbb{Q}})$ of nontrivial torsion points. Then*

$$\lim_{\ell\to+\infty} \mathrm{h}(C \cap \omega_\ell C) = \mathrm{h}_{\overline{\mathscr{O}(1)}^{\mathrm{can}}, \overline{\mathscr{O}(1)}^{\mathrm{Ron}}}(\mathscr{C}).$$

Its proof relies on the next complex analytic equality. As in Section 4 we denote by $\mathbb{S} = (S^1)^2$ the compact torus of $(\mathbb{C}^\times)^2$ and by ν its probability Haar measure.

Lemma 10.5. *Let s_0 be the global section of $\mathscr{O}(1)$ corresponding to the homogeneous coordinate x_0 of $\mathbb{P}^2_{\mathbb{Z}}$. Then*

$$\int_{\mathbb{P}^2(\mathbb{C})} \log \|s_0^{\mathrm{an}}\|_{\mathrm{Ron}}\, \mathrm{c}_1\big(\overline{O(1)}^{\mathrm{can}}\big) \wedge \delta_{\mathscr{C}(\mathbb{C})} = -\int_{\mathbb{S}} \log\max(|z_2 - z_1|, |z_2 - 1|, |z_1 - 1|)\, d\nu(z).$$

Proof. Set for short $\mu = \mathrm{c}_1(\overline{O(1)}^{\mathrm{can}}) \wedge \delta_{\mathscr{C}(\mathbb{C})}$. By construction, it is a measure on $\mathbb{P}^2(\mathbb{C})$ supported on the line $\mathscr{C}(\mathbb{C})$. For each $z \in \mathbb{S} \setminus \{(1,1)\}$ we consider the integral

$$F(z) = \int_{\mathbb{P}^2(\mathbb{C})} \log\Big|\frac{p_0 + z_1 p_1 + z_2 p_2}{p_0}\Big|\, d\mu(p), \tag{10.3}$$

which is finite because the restriction of the integrand to this line is a function which is continuous at all but two points, where it has at most logarithmic singularities.

We will find a simpler expression for this integral as an application of the metric Weil reciprocity law. To this end, consider the inclusion $\jmath\colon \mathscr{C}(\mathbb{C}) \to \mathbb{P}^2(\mathbb{C})$ and the semipositive metrized line bundles on $\mathscr{C}(\mathbb{C})$ given by the inverse image with respect to this map of the trivial and the canonical metrized line bundles on $\mathbb{P}^2(\mathbb{C})$, that is

$$\overline{L_1} = \jmath^*\overline{O_{\mathbb{P}^2(\mathbb{C})}}^{\mathrm{tr}} = \overline{O_{\mathscr{C}(\mathbb{C})}}^{\mathrm{tr}} \quad \text{and} \quad \overline{L_2} = \jmath^*\overline{O(1)}^{\mathrm{can}}.$$

On the one hand, the rational function $s_1 = (x_0+z_1x_1+z_2x_2)/x_0$ of $\mathbb{P}^2(\mathbb{C})$ restricts to a nonzero rational function of $\mathscr{C}(\mathbb{C})$, which gives a nonzero rational section $\jmath^*s_1$ of $L_1 = O_{\mathscr{C}(\mathbb{C})}$. On the other hand, given complex numbers α_1 and α_2 not both equal to 1, the linear polynomial $x_0+\alpha_1x_1+\alpha_2x_2$ corresponds to a global section s_2 of $O(1)$, which restricts to the nonzero global section $\jmath^*s_2$ of $L_2 = \jmath^*O(1)$. The 0-cycles of $\mathscr{C}(\mathbb{C})$ respectively defined by these pulled-back rational sections are

$$Z_1 = [z_1-z_2 : z_2-1 : 1-z_1]-[0:1:-1] \quad \text{and} \quad Z_2 = [\alpha_1-\alpha_2 : \alpha_2-1 : 1-\alpha_1]. \tag{10.4}$$

For an appropriate choice of α_1 and α_2, the supports of these 0-cycles are disjoint.

The restriction of the measure μ to the line coincides with $\mathrm{c}_1\big(\overline{L_2}\big)$, by the functoriality of the Monge–Ampère operator. Moreover $\mathrm{c}_1\big(\overline{L_1}\big) = 0$, as shown in Example 9.2. Hence Proposition 9.6 together with (10.3) and (10.4) implies that

$$\begin{aligned} F(z) &= \int_{\mathscr{C}(\mathbb{C})} \log\|\jmath^*s_1\|_1\,\mathrm{c}_1\big(\overline{L_2}\big) = \int_{\mathscr{C}(\mathbb{C})} \log\|\jmath^*s_1\|_1\,\delta_{Z_2} - \log\|\jmath^*s_2\|_2\,\delta_{Z_1} \\ &= \log\left(\frac{\|s_1([\alpha_1-\alpha_2:\alpha_2-1:1-\alpha_1])\|_{\mathrm{tr}}\cdot\|s_2([0:1:-1])\|_{\mathrm{can}}}{\|s_2([z_1-z_2:z_2-1:1-z_1])\|_{\mathrm{can}}}\right). \end{aligned} \tag{10.5}$$

We have that

$$\begin{aligned} \|s_1([\alpha_1-\alpha_2:\alpha_2-1:1-\alpha_1])\|_{\mathrm{tr}} &= \frac{|(\alpha_1-\alpha_2)+z_1(\alpha_2-1)+z_2(1-\alpha_1)|}{|\alpha_1-\alpha_2|}, \\ \|s_2([0:1:-1])\|_{\mathrm{can}} &= |\alpha_1-\alpha_2|, \\ \|s_2([z_1-z_2:z_2-1:1-z_1])\|_{\mathrm{can}} &= \frac{|(z_1-z_2)+\alpha_1(z_2-1)+\alpha_2(1-z_1)|}{\max(|z_2-z_1|,|z_2-1|,|z_1-1|)}. \end{aligned}$$

Hence (10.5) simplifies to

$$F(z) = \log\max(|z_2-z_1|,|z_2-1|,|z_1-1|), \tag{10.6}$$

in accordance with the notation in (4.1).

We now consider the functions $g,h\colon \mathbb{S}\times\mathbb{P}^2(\mathbb{C}) \to \mathbb{R}\cup\{-\infty\}$ defined as

$$g(z,p) = \log\left|\frac{p_0+z_1p_1+z_2p_2}{p_0}\right| \quad \text{and} \quad h(z,p) = \log\left(\frac{|p_0|+|p_1|+|p_2|}{|p_0|}\right)$$

whenever $(z,p)\notin Z(p_0)$, and as and arbitrary constant otherwise. The subset $Z(p_0)$ has zero mass with respect to the product measure $\nu\times\mu$, and so the integrals of these functions on $\mathbb{S}\times\mathbb{P}^2(\mathbb{C})$ do not depend on the choice of this constant.

The function h is constant with respect to the first variable, whereas, as a function of the second one, its restriction to the line $\mathscr{C}(\mathbb{C})$ only has a logarithmic singularity at a point, making it integrable with respect to $\nu\times\mu$. Moreover $h-g$ is nonnegative, and from (10.3) and (10.6) we deduce that the iterated integral

$$\int_{\mathbb{S}}\int_{\mathbb{P}^2(\mathbb{C})} (h(z,p)-g(z,p))\,d\mu(p)\,d\nu(z)$$

is finite. Tonelli's theorem then implies that the function $h - g$, and *a fortiori* g, is integrable with respect to the product measure $\nu \times \mu$, and that

$$\int_{\mathbb{S}} \int_{\mathbb{P}^2(\mathbb{C})} g(z,p)\, d\mu(p)\, d\nu(z) = \int_{\mathbb{P}^2(\mathbb{C})} \int_{\mathbb{S}} g(z,p)\, d\nu(z)\, d\mu(p). \tag{10.7}$$

Finally note that for $p = [p_0 : p_1 : p_2] \in \mathbb{P}^2(\mathbb{C})$ with $p_0 \neq 0$ we have that

$$\log \|s_0^{\mathrm{an}}(p)\|_{\mathrm{Ron}} = -\int_{\mathbb{S}} g(z,p)\, d\nu(z).$$

It follows from (10.7) that

$$\int_{\mathbb{P}^2(\mathbb{C})} \log \|s_0^{\mathrm{an}}(p)\|_{\mathrm{Ron}}\, d\mu(p) = -\int_{\mathbb{S}} \int_{\mathbb{P}^2(\mathbb{C})} g(z,p)\, d\mu(p)\, d\nu(z) = -\int_{\mathbb{S}} F(z)\, d\nu(z).$$

Together with (10.6), this gives the statement. □

Proof of Theorem 10.4. By Corollary 3.6 and Proposition 4.1,

$$\lim_{\ell \to +\infty} \mathrm{h}(C \cap \omega_\ell C) = \int_{\mathbb{S}} \log \max(|z_2 - z_1|, |z_2 - 1|, |z_1 - 1|)\, d\nu(z).$$

On the other hand, the recursive definition of the Arakelov height shows that

$$\mathrm{h}_{\overline{\mathscr{O}(1)}^{\mathrm{can}}, \overline{\mathscr{O}(1)}^{\mathrm{Ron}}}(\mathscr{C}) = \mathrm{h}_{\overline{\mathscr{O}(1)}^{\mathrm{can}}}(\mathrm{div}(s_0) \cdot \mathscr{C}) - \int_{\mathbb{P}^2(\mathbb{C})} \log \|s_0^{\mathrm{an}}\|_{\mathrm{Ron}}\, \mathrm{c}_1\big(\overline{O(1)}^{\mathrm{can}}\big) \wedge \delta_{\mathscr{C}(\mathbb{C})}.$$

The divisor $\mathrm{div}(s_0) \cdot \mathscr{C}$ coincides with the 1-dimensional irreducible subscheme of $\mathbb{P}^2_{\mathbb{Z}}$ arising as the Zariski closure of the point in the generic fiber of this scheme corresponding to the point $[0 : 1 : -1] \in \mathbb{P}^2(\overline{\mathbb{Q}})$. So (10.1) gives

$$\mathrm{h}_{\overline{\mathscr{O}(1)}^{\mathrm{can}}}(\mathrm{div}(s_0) \cdot \mathscr{C}) = \mathrm{h}([0 : 1 : -1]) = 0.$$

The statement then follows from Lemma 10.5. □

11. A toric perspective

The rest of the article is concerned with expressing and computing the Arakelov height in Theorem 10.4 in terms of convex geometry through the theory of heights of toric varieties and of their hypersurfaces, respectively studied in [BPS14] and in [Gua18b]. Since the only toric variety that we need to consider here is the projective space, we restrict our presentation to this specific case. In spite of that, this is sufficient to give a taste of the general theory.

Let $B \subset \mathbb{R}^n$ be a convex subset and $g\colon B \to \mathbb{R}$ a concave function on it, that is a function satisfying

$$g(\lambda u + (1-\lambda)v) \geq \lambda g(u) + (1-\lambda) g(v) \quad \text{for all } u, v \in B \text{ and } \lambda \in [0,1].$$

Its *stability set* is the subset of $\mathbb{R}^n$ defined as

$$\mathrm{stab}(g) = \big\{ x \in \mathbb{R}^n \;\big|\; \inf_{u \in B} (\langle x, u\rangle - g(u)) > -\infty \big\},$$

where $\langle x, u\rangle = x_1 u_1 + \cdots + x_n u_n$ denotes the usual scalar product. Then the *Legendre–Fenchel dual* of g is the function $g^\vee\colon \mathrm{stab}(g) \to \mathbb{R}$ defined as

$$g^\vee(x) = \inf_{u \in B} (\langle x, u\rangle - g(u)).$$

It is concave and upper semicontinuous. If g is upper semicontinuous, then $g^{\vee\vee} = g$.

Example 11.1. Let Δ_n be the standard simplex of $\mathbb{R}^n$, that is the n-dimensional simplex whose vertices are the origin and the vectors in the standard basis of this vector space. Its *support function* is the function $\Psi_{\Delta_n}\colon \mathbb{R}^n \to \mathbb{R}$ defined as

$$\Psi_{\Delta_n}(u) = \inf_{x\in\Delta_n} \langle x, u\rangle = \min(0, u_1, \dots, u_n).$$

It is piecewise linear and concave, its stability set coincides with Δ_n, and its Legendre–Fenchel dual is 0_{Δ_n}, the zero function on this simplex.

Example 11.2. Let $(S^1)^n$ be the compact torus of $(\mathbb{C}^\times)^n$ and ν_n its probability Haar measure. The *Ronkin function* is the function $\rho_n\colon \mathbb{R}^n \to \mathbb{R}$ defined as

$$\rho_n(u) = -\int_{(S^1)^n} \log|1 + z_1 \mathrm{e}^{-u_1} + \cdots + z_n \mathrm{e}^{-u_n}|\, d\nu_n(z).$$

It is linked to the function R in Example 9.4 by the relation

$$\rho_n(u) = -\log R(1, \mathrm{e}^{-u_1}, \dots, \mathrm{e}^{-u_n}). \tag{11.1}$$

The function ρ_n is concave and its difference with the support function Ψ_{Δ_n} is uniformly bounded [Gua18b, Proposition 2.10]. This implies that its stability set is the standard simplex Δ_n and that its Legendre–Fenchel dual $\rho_n^\vee$ is a continuous concave function on it.

Remark 11.3. Ronkin functions can be associated to any Laurent polynomial, or more generally to any holomorphic function on an open and $(S^1)^n$-invariant subset of $(\mathbb{C}^\times)^n$. As such, they were introduced by Ronkin in [Ron01] and further investigated by Passare and Rullgård in [PR04].

Up to its sign, the function ρ_n in Example 11.2 is the special case corresponding to the Laurent polynomial $1 + t_1 + \cdots + t_n$. For $n = 2$, Cassaigne and Maillot have found an explicit expression for it in terms of the Bloch–Wigner dilogarithm [Mai00, Proposition 7.3.1]. In spite of being our case of interest, we do not need such a description in this article.

The importance of concave functions in our context stems from their interplay with semipositive toric metrics on the hyperplane line bundle of the complex projective space. A metric $\|\cdot\|$ on $O(1)$ is *toric* if

$$\|s_0(zp)\| = \|s_0(p)\| \quad \text{for all } p \in \mathbb{P}^n(\mathbb{C}) \text{ and } z \in (S^1)^n, \tag{11.2}$$

where zp denotes the translation of p by z as in (1.3) and s_0 the global section of $O(1)$ corresponding to the homogeneous coordinate x_0 of $\mathbb{P}^n(\mathbb{C})$.

To a toric metric $\|\cdot\|$ on $O(1)$ one can associate its *metric function*, which is the function $\psi_{\|\cdot\|}\colon \mathbb{R}^n \to \mathbb{R}$ defined as

$$\psi_{\|\cdot\|}(u) = \log \|s_0([1 : \mathrm{e}^{-u_1} : \cdots : \mathrm{e}^{-u_n}])\|.$$

A toric metric on $O(1)$ is semipositive if and only if its metric function is concave [BPS14, Theorem 4.8.1(1)]. When this is the case, the stability set of this concave function is the standard simplex Δ_n, and one also associates to the toric metric its *roof function*, which is the continuous concave function

$$\vartheta_{\|\cdot\|}\colon \Delta_n \longrightarrow \mathbb{R}$$

defined as the Legendre–Fenchel dual of $\psi_{\|\cdot\|}$.

Example 11.4. The canonical metric of $O(1)$ from Example 9.3 is toric, as it can be readily checked from the condition in (11.2), and its metric function coincides with the support function Ψ_{Δ_n} from Example 11.1. This recovers the semipositivity of $\|\cdot\|_{\mathrm{can}}$ and implies that its roof function is the zero function 0_{Δ_n}.

Example 11.5. The Ronkin metric of $O(1)$ from Example 9.4 is also toric by definition, and the associated metric function agrees with the Ronkin function ρ_n from Example 11.2, by means of the relation (11.1). In particular, it is a semipositive metric on $O(1)$ having as roof function its Legendre–Fenchel dual $\rho_n^\vee$.

The main results of [BPS14] and [Gua18b] allow to express the height of $\mathbb{P}^n_{\mathbb{Z}}$ and of its hypersurfaces with respect to a family of semipositive toric metrics on $\mathscr{O}(1)$ in terms of its associated family of roof functions. This is achieved through the *mixed integral* MI, which is an operator on families of $(n+1)$-many continuous concave functions on compact convex subsets of $\mathbb{R}^n$ that is obtained by polarizing the usual integral operator [BPS14, Definition 2.7.16].

Since our case of interest is $\mathbb{P}^2_{\mathbb{Z}}$, we adapt the statements from the adelic setting of the *loc. cit.* to the schematic point of view of the present paper to obtain the following.

Proposition 11.6. *Let $\overline{\mathscr{O}(1)}^{\mathrm{can}}$ and $\overline{\mathscr{O}(1)}^{\mathrm{Ron}}$ be the canonical and Ronkin metrized line bundles of $\mathbb{P}^2_{\mathbb{Z}}$ as in (10.2), and $\mathscr{C}$ the subscheme defined by the linear polynomial $x_0+x_1+x_2$. Then*

$$\mathrm{h}_{\overline{\mathscr{O}(1)}^{\mathrm{can}},\overline{\mathscr{O}(1)}^{\mathrm{Ron}}}(\mathscr{C}) = \mathrm{h}_{\overline{\mathscr{O}(1)}^{\mathrm{can}},\overline{\mathscr{O}(1)}^{\mathrm{Ron}},\overline{\mathscr{O}(1)}^{\mathrm{Ron}}}(\mathbb{P}^2_{\mathbb{Z}}) = \mathrm{MI}(0_{\Delta_2},\rho_2^\vee,\rho_2^\vee).$$

Proof. This proof assumes a working knowledge of the adelic approach to the theory of Arakelov heights as exposed in [BPS14, Chapter 1].

Let $\mathbb{P}^2_{\mathbb{Q}}$ be the projective plane over $\mathrm{Spec}(\mathbb{Q})$ and D the Cartier divisor of the line at infinity. The pair $(\mathbb{P}^2_{\mathbb{Z}},\mathscr{O}(1))$ is an integral model of $(\mathbb{P}^2_{\mathbb{Q}},O(D))$, and so the canonical and the Ronkin metrized line bundles on $\mathbb{P}^2_{\mathbb{Z}}$ induce quasi-algebraic semipositive adelic metrized divisors

$$\overline{D}^{\mathrm{can}} \qquad \text{and} \qquad \overline{D}^{\mathrm{Ron}}$$

on $\mathbb{P}^2_{\mathbb{Q}}$ in the sense of [BPS14, Section 1.5]. Namely, their Archimedean metrics coincide respectively with those of $\overline{\mathscr{O}(1)}^{\mathrm{can}}$ and $\overline{\mathscr{O}(1)}^{\mathrm{Ron}}$, and so do their Archimedean roof functions. Instead, their non-Archimedean metrics are those induced by the integral model $(\mathbb{P}^2_{\mathbb{Z}},\mathscr{O}(1))$ as in [BPS14, Example 1.3.11], and then their roof functions agree with the zero function on Δ_n.

This correspondence is compatible with the associated heights, and thus [BPS14, Theorem 5.2.5] together with Examples 11.4 and 11.5 specializes to

$$\mathrm{h}_{\overline{\mathscr{O}(1)}^{\mathrm{can}},\overline{\mathscr{O}(1)}^{\mathrm{Ron}},\overline{\mathscr{O}(1)}^{\mathrm{Ron}}}(\mathbb{P}^2_{\mathbb{Z}}) = \mathrm{h}_{\overline{D}^{\mathrm{can}},\overline{D}^{\mathrm{Ron}},\overline{D}^{\mathrm{Ron}}}(\mathbb{P}^2_{\mathbb{Q}}) = \mathrm{MI}(0_{\Delta_2},\rho_2^\vee,\rho_2^\vee),$$

which gives the second equality in the statement. We also have that

$$\mathrm{h}_{\overline{\mathscr{O}(1)}^{\mathrm{can}},\overline{\mathscr{O}(1)}^{\mathrm{Ron}}}(\mathscr{C}) = \mathrm{h}_{\overline{D}^{\mathrm{can}},\overline{D}^{\mathrm{Ron}}}(\mathscr{C}_{\mathbb{Q}})$$

because $\mathscr{C}$ contains no vertical component, and [Gua18b, Theorem 5.12] gives

$$\mathrm{h}_{\overline{D}^{\mathrm{can}},\overline{D}^{\mathrm{Ron}}}(\mathscr{C}_{\mathbb{Q}}) = \mathrm{h}_{\overline{D}^{\mathrm{can}},\overline{D}^{\mathrm{Ron}},\overline{D}^{\mathrm{Ron}}}(\mathbb{P}^2_{\mathbb{Q}})$$

as $\overline{D}^{\mathrm{Ron}}$ coincides with the Ronkin metrized divisor of the Laurent polynomial $1+t_1+t_2$ in the sense of [Gua18b]. This gives the first equality and completes the proof. $\square$

Combining this result with Theorem 10.4 and the equality in (10.1) yields the following statement.

Corollary 11.7. *Let $(\omega_\ell)_{\ell\geq 1}$ be a strict sequence in $\mathbb{G}_{\mathrm{m}}^2(\overline{\mathbb{Q}})$ of nontrivial torsion points. Then*

$$\lim_{\ell\to+\infty} \mathrm{h}_{\overline{\mathscr{O}(1)}^{\mathrm{can}}}(Z(x_0+x_1+x_2,\, x_0+\omega_{\ell,1}^{-1}x_1+\omega_{\ell,2}^{-1}x_2)) = \mathrm{h}_{\overline{\mathscr{O}(1)}^{\mathrm{can}},\,\overline{\mathscr{O}(1)}^{\mathrm{Ron}},\,\overline{\mathscr{O}(1)}^{\mathrm{Ron}}}(\mathbb{P}^2_{\mathbb{Z}}),$$

where $Z(x_0+x_1+x_2, x_0+\omega_{\ell,1}^{-1}x_1+\omega_{\ell,2}^{-1}x_2)$ denotes the 1-dimensional subscheme of $\mathbb{P}^2_{\mathbb{Z}}$ defined by these linear forms.

This limit formula is a particular case of a conjectural arithmetic analogue of the geometric fact that for a family of n-many line bundles on an n-dimensional algebraic variety, the cardinality of the zero set of a generic n-tuple of their global sections coincides with the degree of the variety with respect to these line bundles. Formulating this conjecture requires the language of adelic metrized line bundles on varieties over global fields, which would take us too far away from our setting. Instead, we content ourselves by explaining a particular case that can be expressed with the objects at hand and still gives a hint of the general case.

For $d \geq 1$, consider the d-tensor power line bundle $O(d) = O(1)^{\otimes d}$ on $\mathbb{P}^n(\mathbb{C})$. Its global sections are in one-to-one correspondence with the homogeneous polynomials of degree d in the homogeneous coordinates of this projective space. Extending Example 9.4, we define the *Ronkin metric* of $O(d)$ setting, for each global section s,

$$\|s(p)\|_{\mathrm{Ron}} = \frac{|l_s(p_0,\dots,p_n)|}{R(|p_0|^d,\dots,|p_n|^d)} \qquad \text{for all } p=[p_0:\cdots:p_n]\in\mathbb{P}^n(\mathbb{C}),$$

where l_s is the homogeneous polynomial of degree d corresponding to s, and R is the same function as that of the referred example. This metric is toric and semipositive, and the corresponding metrized line bundle on $\mathbb{P}^2_{\mathbb{Z}}$ is denoted by $\overline{\mathscr{O}(d)}^{\mathrm{Ron}}$.

Conjecture 11.8. *Let $(\omega_\ell)_{\ell\geq 1}$ be a strict sequence in $\mathbb{G}_{\mathrm{m}}^2(\overline{\mathbb{Q}})$ of nontrivial torsion points. Then for $d_1, d_2 \geq 1$ we have that*

$$\lim_{\ell\to+\infty} \mathrm{h}_{\overline{\mathscr{O}(1)}^{\mathrm{can}}}(Z(x_0^{d_1}+x_1^{d_1}+x_2^{d_1},\, x_0^{d_2}+\omega_{\ell,1}^{-d_2}x_1^{d_2}+\omega_{\ell,2}^{-d_2}x_2^{d_2})) \\ = \mathrm{h}_{\overline{\mathscr{O}(1)}^{\mathrm{can}},\,\overline{\mathscr{O}(d_1)}^{\mathrm{Ron}},\,\overline{\mathscr{O}(d_2)}^{\mathrm{Ron}}}(\mathbb{P}^2_{\mathbb{Z}}).$$

In contrast with Corollary 11.7, this statement would allow to predict this kind of limit heights in situations where the considered systems of equations cannot be explicitly solved.

12. Integrals over an amoeba

In Section 4 we considered the co-tropicalization map on $(\mathbb{C}^\times)^2$ that sends each point to the arguments of its coordinates. Looking instead at their absolute values gives the more classical *tropicalization map*:

$$\mathrm{trop}\colon (\mathbb{C}^\times)^2 \longrightarrow \mathbb{R}^2, \qquad z \longmapsto (-\log|z_1|, -\log|z_2|).$$

The image of a curve of $(\mathbb{C}^\times)^2$ under this map is called its *amoeba*. The term was coined in [GKZ08, Section 6.1], where these tentacle-shaped subsets were introduced.

Here we will be concerned with the amoeba of the curve of $(\mathbb{C}^\times)^2$ defined by the Laurent polynomial $1+t_1+t_2$, that is

$$\mathscr{A} = \{(-\log|z_1|, -\log|z_2|) \mid \gamma\in(\mathbb{C}^\times)^2 \text{ such that } 1+z_1+z_2=0\}. \tag{12.1}$$

We will refer to this subset of $\mathbb{R}^2$ as the *Archimedean amoeba* of C, the line of $\mathbb{P}^2(\overline{\mathbb{Q}})$ studied throughout Part I. It is depicted in Figure 12.1, and its computation is explained in [Gua20].

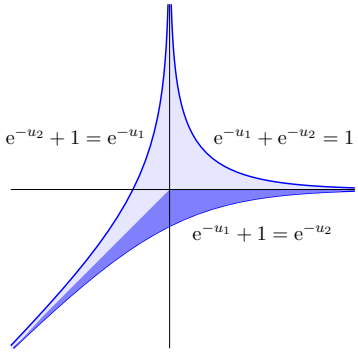

FIGURE 12.1. The Archimedean amoeba of C with its contour lines and south region

Remark 12.1. This terminology can be justified as follows. The line $C \subset \mathbb{P}^2(\overline{\mathbb{Q}})$ is defined over the rationals, and so it can be identified with an integral subscheme of $\mathbb{P}^2_{\mathbb{Q}}$ whose Archimedean analytification is the complex line $Z(x_0+x_1+x_2) \subset \mathbb{P}^2(\mathbb{C})$. Then the subset $\mathscr{A}$ coincides with the amoeba of the restriction of this complex line to the dense open subset $\mathbb{P}^2(\mathbb{C}) \setminus Z(x_0x_1x_2) \simeq (\mathbb{C}^\times)^2$.

For the sequel, consider the subset of this Archimedean amoeba given as

$$\mathscr{A}_{\text{south}} = \{u \in \mathscr{A} \mid \min(0, u_1) \geq u_2\},$$

which is the region colored in dark blue in Figure 12.1. The next result shows that the integrals on this region of the powers of a coordinate function are given by special values of the Riemann zeta function.

Proposition 12.2. *For all $m \in \mathbb{N}$ we have that*

$$\int_{\mathscr{A}_{\text{south}}} u_2^m \, du_1 du_2 = (-1)^m \, m! \, \zeta(m+2).$$

Proof. The nontrivial boundary of the region $\mathscr{A}_{\text{south}}$ is given by $u_1 = -\log(e^{-u_2}-1)$, as indicated in Figure 12.1. Then Tonelli's theorem and a direct computation yield

$$\begin{aligned} \int_{\mathscr{A}_{\text{south}}} u_2^m \, du_1 du_2 &= \int_{-\infty}^{0} \left(\int_{u_2}^{-\log(e^{-u_2}-1)} du_1 \right) u_2^m \, du_2 \\ &= \int_{-\infty}^{0} u_2^m (-\log(e^{-u_2}-1) - u_2) \, du_2 = \int_{-\infty}^{0} -u_2^m \log(1-e^{u_2}) \, du_2. \end{aligned} \tag{12.2}$$

The absolute convergence of the series for the logarithm on the open unit disk and the obvious statement for $u_2 = 0$ imply the pointwise convergence

$$-u_2^m \log(1-e^{u_2}) = \sum_{k=1}^{\infty} \frac{u_2^m}{k} e^{ku_2} \quad \text{for } u_2 \in (-\infty, 0],$$

as functions with values in $\mathbb{R} \cup \{\pm\infty\}$. Since every summand of the series has the same sign, Levi's monotone convergence theorem implies that

$$\int_{-\infty}^{0} -u_2^m \log(1-\mathrm{e}^{u_2})\, du_2 = \int_{-\infty}^{0} \sum_{k=1}^{\infty} \frac{u_2^m}{k} \mathrm{e}^{ku_2}\, du_2 = \sum_{k=1}^{\infty} \frac{1}{k} \int_{-\infty}^{0} u_2^m \mathrm{e}^{ku_2}\, du_2 \tag{12.3}$$
$$= \sum_{k=1}^{\infty} \frac{1}{k} \left[\mathrm{e}^{ku_2} \sum_{\ell=0}^{m} \frac{(-1)^\ell m!}{k^{\ell+1}(m-\ell)!} u_2^{m-\ell} \right]_{-\infty}^{0}$$
$$= \sum_{k=1}^{\infty} (-1)^m \frac{m!}{k^{m+2}} = (-1)^m\, m!\, \zeta(m+2).$$

The statement follows from (12.2) and (12.3). □

This result can be employed to recover the computation of the area of $\mathscr{A}$ by Passare and Rullgård in [PR04, page 502]. A similar computation appears in [KP16, page 919].

Example 12.3. For $m = 0$, Proposition 12.2 gives that $\mathrm{vol}(\mathscr{A}_{\mathrm{south}}) = \zeta(2) = \pi^2/6$. Note that $\mathscr{A}$ is disjointly covered, up to subsets of measure zero, by the images of the south region under the map $(u_1, u_2) \mapsto (-u_2, u_1 - u_2)$ and under the symmetry along the diagonal. Since these maps preserve the Lebesgue measure, we deduce that

$$\mathrm{vol}(\mathscr{A}) = 3\ \mathrm{vol}(\mathscr{A}_{\mathrm{south}}) = 3\zeta(2) = \frac{\pi^2}{2}.$$

The previous proposition also allows to compute the integral on the Archimedean amoeba of C of the support function of the 2-dimensional standard simplex.

Example 12.4. For $m = 1$, Proposition 12.2 gives that $\int_{\mathscr{A}_{\mathrm{south}}} u_2\, du_1 du_2 = -\zeta(3)$. Consider the subsets of $\mathscr{A}$ defined as

$$\mathscr{A}_{\mathrm{east}} = \{u \in \mathscr{A} \mid \min(u_1, u_2) \geq 0\} \quad \text{and} \quad \mathscr{A}_{\mathrm{west}} = \{u \in \mathscr{A} \mid \min(0, u_2) \geq u_1\}.$$

Then

$$\int_{\mathscr{A}} \min(0, u_1, u_2)\, du_1 du_2 = \int_{\mathscr{A}_{\mathrm{east}}} 0\, du_1 du_2 + \int_{\mathscr{A}_{\mathrm{west}}} u_1\, du_1 du_2 + \int_{\mathscr{A}_{\mathrm{south}}} u_2\, du_1 du_2$$
$$= 2 \int_{\mathscr{A}_{\mathrm{south}}} u_2\, du_1 du_2 = -2\,\zeta(3),$$

where the second equality follows from the symmetries of this integral.

Remark 12.5. It would be interesting to explore if there are other integrals of piecewise polynomial functions on amoebas of curves that can be expressed in terms of special values of L-functions.

13. Computing a mixed integral

Finally, we show how the mixed integral in Proposition 11.6 relates to the integral of a piecewise linear function on the amoeba treated in Section 12. The obtained equality offers a further expression for the limit height under study, and it can be used to recover our main result.

Set $\Delta = \Delta_2$ for the standard simplex of $\mathbb{R}^2$ and denote by 0_Δ the zero function on it. Set also $\Psi = \Psi_{\Delta_2}$ for the support function of this simplex as in Example 11.1, and $\rho = \rho_2$ for the Ronkin function of $1 + t_1 + t_2$ according to Example 11.2. Recall that $\mathscr{A}$ denotes the Archimedean amoeba of the line C as in (12.1).

Theorem 13.1. *With notation as above, we have that*

$$\mathrm{MI}(0_\Delta, \rho^\vee, \rho^\vee) = -\frac{1}{\mathrm{vol}(\mathscr{A})} \int_{\mathscr{A}} \Psi(u)\, du_1 du_2,$$

where vol *denotes the Lebesgue measure of* $\mathbb{R}^2$.

To prove this result, we first need to understand the behavior of the Legendre–Fenchel dual of this Ronkin function on the boundary of its domain.

Lemma 13.2. *The function* $\rho^\vee$ *vanishes on the boundary of* Δ.

Proof. It is a consequence of Jensen's formula that

$$\rho(u) = \Psi(u) \quad \text{for all } u \in \mathbb{R}^2 \setminus \mathscr{A}, \tag{13.1}$$

see for instance [Mai00, Proposition 7.3.1(2)]. Since ρ is concave, this implies that the inequality $\rho \leq \Psi$ holds on the entire real plane. Taking Legendre–Fenchel duals we obtain that $\rho^\vee(x) \geq 0_\Delta(x) = 0$ for all $x \in \Delta$.

Hence, to complete the proof it is enough to show that $\rho^\vee$ is nonpositive on the boundary of Δ. To this aim, first consider a point $x = (\lambda, 0) \in \partial\Delta$ with $\lambda \in [0,1]$. For $\varepsilon > 0$ the point $u_\varepsilon = (\varepsilon, -\log(1 - \mathrm{e}^{-\varepsilon}))$ belongs to the east border of $\mathscr{A}$, as shown in Figure 12.1. By (13.1) and the continuity of the Ronkin function we necessarily have that $\rho(u_\varepsilon) = 0$. Hence

$$\rho^\vee(x) = \inf_{u \in \mathbb{R}^2} (\langle x, u\rangle - \rho(u)) \leq \lim_{\varepsilon \to 0^+} (\langle x, u_\varepsilon\rangle - \rho(u_\varepsilon)) = \lim_{\varepsilon \to 0^+} \lambda\varepsilon = 0,$$

which implies that $\rho^\vee(x) = 0$. The fact that this also holds when $x = (0, \lambda)$ with $\lambda \in [0,1]$ can be proven similarly.

To conclude, let $x = (\lambda, 1-\lambda)$ with $\lambda \in [0,1]$. For $\kappa > 0$ consider the point $u_\kappa = (-\log(\mathrm{e}^\kappa - 1), -\kappa)$, which lies in the southern border of $\mathscr{A}$. Again by (13.1) and the continuity of the Ronkin function we have that $\rho(u_\kappa) = -\kappa$. Hence

$$\begin{aligned}\rho^\vee(x) \leq \lim_{\kappa \to +\infty} (\langle (x, u_\kappa\rangle - \rho(u_\kappa)) &= \lim_{\kappa \to +\infty} (\lambda \cdot (-\log(\mathrm{e}^\kappa - 1)) + (1-\lambda)\cdot(-\kappa) - (-\kappa)) \\ &= \lim_{\kappa \to +\infty} -\lambda \log(1 - \mathrm{e}^{-\kappa}) = 0,\end{aligned}$$

which implies that $\rho^\vee(x) = 0$ also holds in this case, as stated. □

We can now move to the proof of Theorem 13.1, which relies on a relation between the mixed integral and an integral on the dual space with respect to the real Monge–Ampère measure of a concave function. The latter is a positive measure having higher density where the function is more concave [BPS14, Definition 2.7.1].

Proof of Theorem 13.1. Since both 0_Δ and $\rho^\vee$ are continuous concave functions on the simplex Δ of $\mathbb{R}^2$, it is a consequence of [Gua18b, Theorem 1.6] together with the vanishing of $\rho^\vee$ on the boundary of Δ shown in Lemma 13.2 that

$$\mathrm{MI}(0_\Delta, \rho^\vee, \rho^\vee) = -2 \int_{\mathbb{R}^2} \Psi \, d\mathcal{M}(\rho), \tag{13.2}$$

where $\mathcal{M}(\rho)$ stands for the real Monge–Ampère measure of the Ronkin function.

By [PR04, Theorem 7 and Example on page 502] the real Monge–Ampère measure of ρ agrees with the Lebesgue measure of $\mathbb{R}^2$ restricted to the amoeba $\mathscr{A}$, and scaled by the constant π^{-2}, and so

$$\int_{\mathbb{R}^2} \Psi \, d\mathcal{M}(\rho) = \frac{1}{\pi^2} \int_{\mathscr{A}} \Psi(u)\, du_1 du_2. \tag{13.3}$$

The statement follows from (13.2) and (13.3) together with Example 12.3. □

The following corollary is an immediate consequence of the chain of equalities given by Theorem 10.4, Proposition 11.6 and Theorem 13.1.

Corollary 13.3. *Let $(\omega_\ell)_{\ell\geq 1}$ be a strict sequence in $\mathbb{G}_{\mathrm{m}}^2(\overline{\mathbb{Q}})$ of nontrivial torsion points. Then*

$$\lim_{\ell\to+\infty} \mathrm{h}(C\cap\omega_\ell C) = \mathrm{h}_{\overline{\mathscr{O}(1)}^{\mathrm{can}},\,\overline{\mathscr{O}(1)}^{\mathrm{Ron}}}(\mathscr{C}) = -\frac{1}{\mathrm{vol}(\mathscr{A})}\int_{\mathscr{A}} \Psi(u)\,du_1du_2,$$

where $\mathscr{C}$ is the subscheme of $\mathbb{P}^2_{\mathbb{Z}}$ defined by $x_0+x_1+x_2$, whereas $\overline{\mathscr{O}(1)}^{\mathrm{can}}$ and $\overline{\mathscr{O}(1)}^{\mathrm{Ron}}$ are the canonical and Ronkin semipositive metrized line bundles on $\mathbb{P}^2_{\mathbb{Z}}$.

This result highlights the role played by our particular choice of both height function and curve. Indeed, the considered limit height turns out to be computed by the average of Ψ, which is metric function associated to the canonical height in the toric correspondence from [BPS14], on the Archimedean amoeba $\mathscr{A}$ of the line C.

Finally, the right-hand side in the equalities in Corollary 13.3 can be directly computed using Examples 12.3 and 12.4, namely

$$-\frac{1}{\mathrm{vol}(\mathscr{A})}\int_{\mathscr{A}} \Psi(u)\,du_1du_2 = \frac{2\,\zeta(3)}{3\,\zeta(2)},$$

thus recovering Theorem 6.1 through this more conceptual point of view.

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